LOGICS FOR COMPUTER SCIENCE: Classical and Non-Classical

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Chapter 10
Predicate Automated Proof Systems
Completeness of Classical Predicate Logic

CHAPTER 10 SLIDES

Chapter 10
Predicate Automated Proof Systems
Completeness of Classical Predicate Logic

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Chapter 10
Predicate Automated Proof Systems
Completeness of Classical Predicate Logic

Slides Set 1

PART 1: **QRS** Proof System

Predicate Automated Proof Systems
Introduction

We define and discuss here Rasiowa and Sikorski Gentzen style proof system **QRS** for classical predicate logic

The propositional version of it, the **RS** proof system, was studied in detail in chapter 6

These both proof systems **RS** and **QRS** admit a constructive proof of **completeness** theorem

Predicate Automated Proof Systems
Introduction

We adopt Rasiowa, Sikorski (1961) technique of construction a counter model determined by a decomposition tree to prove **QRS** completeness theorem

The proof, presented here is a generalization of the completeness proofs of **RS** and other Gentzen style propositional systems presented in details in chapter 6.

We refer the reader to the chapter 6 as it provides a good introduction to the subject

Predicate Automated Proof Systems
Introduction

The other Gentzen type predicate proof system, including the original Gentzen proof systems **LK, LI** for classical and intuitionistic predicate logics are obtained from their propositional versions discussed in detail in chapter 6 by adding the Quantifiers Rules to them

It can be done in a similar way as a generalization of the propositional **RS** to the the predicate **QRS** system presented here

We leave these generalizations as an exercises for the reader

Predicate Automated Proof Systems
Introduction

We also leave as an exercise the predicate language version of Gentzen proof of cut elimination theorem, Hauptzatz (1935)

The Hauptzatz proof for the predicate classical **LK** and intuitionistic **LI** systems is easily obtained from the propositional proof included in chapter 6

There are of course other types of automated proof systems based on different methods of deduction

Predicate Automated Proof Systems
Introduction

There is a **Natural Deduction** mentioned by Gentzen in his Hauptzatz paper in 1935

It was later and fully developed by Dag Prawitz 1965)

It is now called Prawitz, or Gentzen-Prawitz Natural Deduction

There is a **Semantic Tableaux** deduction method invented by Evert Beth (1955)

It was consequently simplified and further developed by Raymond Smullyan (1968)

It is now often called Smullyan Semantic Tableaux

Finally, there is **Resolution**

The resolution method can be traced back to Davis and Putnam (1960)

Their work is still known as Davis-Putnam method

The difficulties of Davis-Putnam method were eliminated by John Alan Robinson (1965)

He consequently developed it into what we call now Robinson Resolution, or just **Resolution**

Predicate Automated Proof Systems
Introduction

The resolution proof system for propositional or predicate logic operates on a set of clauses as a basic expressions and uses a resolution rule as the only rule of inference

We define and prove **correctness** of effective procedures of converting any formula A into a corresponding set of clauses in both propositional and predicate cases

QRS Proof System

QRS Proof System

The components of the proof system **QRS** are as follows

Language $\mathcal{L}$

$$\mathcal{L} = \mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg\}}(\mathbf{P}, \mathbf{F}, \mathbf{C})$$

for **P, F, C** countably infinite sets of predicate, functional, and constant symbols respectively

Expressions $\mathcal{E}$

Let $\mathcal{F}$ denote a set of formulas of $\mathcal{L}$. We adopt as the set of expressions the set of all finite sequences of formulas, i.e.

$$\mathcal{E} = \mathcal{F}^*$$

We will denote the expressions of **QRS**, i.e. the finite sequences of formulas by

$\Gamma, \quad \Delta, \quad \Sigma,$ with indices if necessary

Rules of Inference of **QRS**

The system **QRS** consists of two axiom schemas and eleven rules of inference

The rules of inference form **two groups**

First group is similar to the propositional case and contains propositional connectives rules:

$$(\cup), \quad (\neg\cup), \quad (\cap), \quad (\neg\cap), \quad (\Rightarrow), \quad (\neg \Rightarrow), \quad (\neg\neg)$$

Second group deals with the quantifiers and consists of four rules:

$$(\forall), \quad (\exists), \quad (\neg\forall), \quad (\neg\exists)$$

Logical Axioms of **RS**

We adopt as logical axioms of **QRS** any sequence of formulas which contains a formula and its negation, i.e any sequence

$$\Gamma_1,\ A,\ \Gamma_2,\ \neg A,\ \Gamma_3$$

$$\Gamma_1,\ \neg A,\ \Gamma_2,\ A,\ \Gamma_3$$

where $A \in \mathcal{F}$ is any **formula**

We denote by LA the set of all logical axioms of **QRS**

Proof System **QRS**

Formally we define the system **QRS** as follows

$$\mathbf{QRS} = (\mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg\}}(\mathbf{P},\mathbf{F},\mathbf{C}),\ \ \mathcal{F}^*,\ \ LA,\ \ \mathcal{R})$$

where the set $\mathcal{R}$ of inference rules contains the following rule

$(\cup)$, $(\neg\cup)$, $(\cap)$, $(\neg\cap)$, $(\Rightarrow)$, $(\neg\Rightarrow)$, $(\neg\neg)$, $(\forall)$, $(\exists)$, $(\neg\forall)$, $(\neg\exists)$

and LA is the set of all logical axioms defined on previous slide

Literals in **QRS**

Definition

Any atomic formula, or a negation of atomic formula is called a literal

We form, as in the propositional case, a special subset

$$LT \subseteq \mathcal{F}$$

of formulas, called a **set of all literals** defined now as follows

$$LT = \{A \in \mathcal{F} : A \in A\mathcal{F}\} \cup \{\neg A \in \mathcal{F} : A \in A\mathcal{F}\}$$

The elements of the set $\{A \in \mathcal{F} : A \in A\mathcal{F}\}$ are called **positive literals**

The elements of the set $\{\neg A \in \mathcal{F} : A \in A\mathcal{F}\}$ are called **negative literals**

Sequences of Literals

We denote by

$$\Gamma', \quad \Delta', \quad \Sigma' \dots$$

finite sequences (empty included) formed out of **literals** i.e

$$\Gamma', \Delta', \Sigma' \in LT^*$$

We will denote by

$$\Gamma, \quad \Delta, \quad \Sigma \dots$$

the elements of $\mathcal{F}^*$

Connectives Inference Rules of **QRS**

Group 1

Disjunction rules

$$(\cup)\ \frac{\Gamma',\ A, B,\ \Delta}{\Gamma',\ (A \cup B),\ \Delta} \qquad (\neg\cup)\ \frac{\Gamma',\ \neg A,\ \Delta \quad ; \quad \Gamma',\ \neg B,\ \Delta}{\Gamma',\ \neg(A \cup B),\ \Delta}$$

Conjunction rules

$$(\cap)\ \frac{\Gamma',\ A,\ \Delta \quad ; \quad \Gamma',\ B,\ \Delta}{\Gamma',\ (A \cap B),\ \Delta} \qquad (\neg\cap)\ \frac{\Gamma',\ \neg A,\ \neg B,\ \Delta}{\Gamma',\ \neg(A \cap B),\ \Delta}$$

where $\Gamma' \in LT^*,\ \ \Delta \in \mathcal{F}^*,\ \ A, B \in \mathcal{F}$

Connectives Inference Rules of **QRS**

Group 1

Implication rules

$$(\Rightarrow)\ \frac{\Gamma',\ \neg A, B,\ \Delta}{\Gamma',\ (A \Rightarrow B),\ \Delta} \qquad (\neg \Rightarrow)\ \frac{\Gamma',\ A,\ \Delta \quad : \quad \Gamma',\ \neg B,\ \Delta}{\Gamma',\ \neg(A \Rightarrow B),\ \Delta}$$

Negation rule

$$(\neg\neg)\ \frac{\Gamma',\ A,\ \Delta}{\Gamma',\ \neg\neg A,\ \Delta}$$

where $\Gamma' \in LT^*,\ \ \Delta \in \mathcal{F}^*,\ \ A, B \in \mathcal{F}$

Quantifiers Inference Rules of **QRS**

Group 2: Universal Quantifier rules

$$(\forall)\ \frac{\Gamma',\ A(y),\ \Delta}{\Gamma',\ \forall x A(x),\ \Delta} \qquad\qquad (\neg\forall)\ \frac{\Gamma',\ \neg\forall x A(x),\ \Delta}{\Gamma',\ \exists x \neg A(x),\ \Delta}$$

where $\Gamma' \in LT^*,\ \ \Delta \in \mathcal{F}^*,\ \ A, B \in \mathcal{F}$

The variable y in rule $(\forall)$ is a free individual variable which **does not** appear in any formula in the conclusion, i.e. in any formula in the sequence $\Gamma', \forall x A(x), \Delta$

The variable y in the rule $(\forall)$ is called the eigenvariable

All occurrences] of y in A(y) of the rule $(\forall)$ are fully indicated

Quantifiers Inference Rules of **QRS**

Group 2: **Existential Quantifier rules**

$$(\exists)\ \frac{\Gamma',\ A(t),\ \Delta, \exists x A(x)}{\Gamma',\ \exists x A(x),\ \Delta} \qquad\qquad (\neg\exists)\ \frac{\Gamma',\ \neg\exists x A(x),\ \Delta}{\Gamma',\ \forall x \neg A(x),\ \Delta}$$

where $t \in T$ is an arbitrary term, $\Gamma' \in LT^*$, $\Delta \in \mathcal{F}^*$, $A, B \in \mathcal{F}$

Note that $A(t), A(y)$ denotes a formula obtained from $A(x)$ by writing the term t or y, respectively, in place of all occurrences of x in A

Proofs and Proof Trees

By a **formal proof** of a sequence Γ in the proof system **QRS** we understand any sequence

$$\Gamma_1,\ \Gamma_2, ...\ \Gamma_n$$

of sequences of formulas (elements of $\mathcal{F}^*$), such that

1. $\Gamma_1 \in LA,\ \Gamma_n = \Gamma$, and

2. for all i $(1 \leq i \leq n)$, $\Gamma_i \in LA$, or Γ_i is a conclusion of one of the inference rules of **QRS** with all its premisses placed in the sequence $\Gamma_1,\ \Gamma_2,\ ....\ \Gamma_{i-1}$

Proofs and Proof Trees

We write, as usual,

$$\vdash_{QRS} \Gamma$$

to denote that the sequence Γ has a formal proof in **QRS**

As the proofs in **QRS** are sequences (definition of the formal proof) of sequences of formulas (definition of expressions $\mathcal{E}$) we will not use " ; " to separate the steps of the proof, and write the formal proof as

$$\Gamma_1; \; \Gamma_2; \; .... \; \Gamma_n$$

Proofs and Proof Trees

We write, however, the formal proofs in **QRS** as we did the propositional case (chapter 6), in a form of **trees** rather then in a form of sequences

We adopt hence the following definition

Proof Tree

By a proof tree, or **QRS** - tree proof of Γ we understand a tree $\mathbf{T}_{\Gamma}$ of sequences satisfying the following conditions:

1. The topmost sequence, i.e the **root** of $\mathbf{T}_{\Gamma}$ is Γ,

2. all **leafs** are axioms,

3. the **nodes** are sequences such that each sequence on the tree follows from the ones immediately preceding it by one of the rules of inference rules

Proof Trees

We picture, and write the proof trees with the **root** on the top, and **leafs** on the very bottom

In particular cases, as in the propositional case, we write the proof trees indicating additionally the **name** of the inference rule used at each step of the proof

For example, when in a proof of a formula A we use subsequently the rules

$$(\cap),\ (\cup),\ (\forall),\ (\cap),\ (\neg\neg),\ (\forall),\ (\Rightarrow)$$

we represent the proof of A as the following tree

Proof Trees

$\mathbf{T}_A$

Formula A

$\mid (\Rightarrow)$

conclusion of $(\forall)$

$\mid (\forall)$

conclusion of $(\neg\neg)$

$\mid (\neg\neg)$

conclusion of $(\cap)$

$\bigwedge (\cap)$

conclusion of $(\forall)$ | *conclusion of* $(\cup)$

$\mid (\forall)$ | $\mid (\cup)$

axiom | *conclusion of* $(\cap)$

$\bigwedge (\cap)$

axiom *axiom*

DecompositionTrees

The main advantage of the Gentzen type proof systems lies in the way we are able to search for proofs in them

Moreover, such proof search happens to be **deterministic** and **automatic**

We conduct **proof search** by treating inference rules as decomposition rules (see chapter 6) and by building decomposition trees

A general principle of building decomposition trees is the following.

Decomposition Tree $\mathbf{T}_{\Gamma}$

For each $\Gamma \in \mathcal{F}^*$, a decomposition tree $\mathbf{T}_{\Gamma}$ is a tree build as follows

Step 1. The sequence Γ is the **root** of $\mathbf{T}_{\Gamma}$

For any node Δ of the tree we follow the steps bellow

Step 2. If Δ is **indecomposable** or an **axiom**, then Δ becomes a **leaf** of the tree

Decomposition Trees

Step 3. If Δ is **decomposable**, then we traverse Δ from left to right to identify the first **decomposable formula** B and identify inference rule treated as decomposition rule that is determined uniquely by B

We put its left and right premisses as the left and right **leaves**, respectively

Step 4. We repeat steps **2.** and **3.** until we obtain only **leaves** or an **infinite branch**

In particular case when when Γ has only one element, namely a a formula $A \in \mathcal{F}$, we call it a decomposition tree of A and denote by $\mathbf{T}_A$

QRS Decomposition Trees

Given a formula $A \in \mathcal{F}$, we define its **decomposition tree** $\mathbf{T}_A$ as follows

Observe that the inference rules of **QRS** can be divided in two groups: propositional connectives rules

$$(\cup), (\neg\cup), (\cap), (\neg\cap), (\Rightarrow), (\neg \Rightarrow)$$

and quantifiers rules

$$(\forall), \quad (\exists), \quad (\neg\forall) \quad (\neg\exists)$$

We define the **decomposition tree** in the case of the propositional rules and the quantifiers rules $(\neg\forall)$, $(\neg\exists)$ in the same way as for the propositional language (chapter 6)

QRS Decomposition Trees

The case of the rules $(\forall)$ and $(\exists)$ is more complicated, as the rules contain the **specific conditions** under which they are applicable

To define the way of **decomposing** the sequences of the form

$$\Gamma^{'}, \forall x A(x), \Delta \quad \text{or} \quad \Gamma^{'}, \exists x A(x), \Delta,$$

i.e. to deal with the rules quantifiers rules $(\forall)$ and $(\exists)$ **we assume** that **all terms** form a one-to one sequence

$$ST \qquad t_1, t_2, ..., t_n, ...$$

Observe, that by the definition, all free variables are terms, hence all free variables **appear** in the sequence ST of all terms

QRS Decomposition Trees

Let Γ be a sequence on the tree in which the first indecomposable formula has the quantifier $\forall$ as its main connective. It means that Γ is of the form

$$\Gamma^{'}, \ \forall_x A(x), \ \Delta$$

We write a sequence

$$\Gamma^{'}, \ A(y), \ \Delta$$

below Γ on the tree as its **child**, where the variable y fulfills the following condition

Condition 1 : the variable y is the **first** free variable in the sequence ST of terms such that y **does not** appear in any formula in $\Gamma^{'}, \forall x A(x), \Delta$

Observe, that the condition the **Condition 1** corresponds to the restriction put on the application of the rule $(\forall)$

QRS Decomposition Trees

Let now the first indecomposable formula in Γ has the quantifier $\exists$ as its main connective. It means that Γ is of the form

$$\Gamma', \ \exists x A(x), \ \Delta$$

We write a sequence

$$\Gamma', \ A(t), \ \Delta, \exists x A(x)$$

as its **child**, where the term t fulfills the following condition

Condition 2: the term t is the first term in the sequence ST of all terms such that the formula A(t) **does not** appear in any sequence on the tree which is placed **above**

$$\Gamma', A(t), \Delta, \exists x A(x)$$

QRS Decomposition Trees

Observe that the sequence ST of all terms is one- to - one and by the **Condition 1** and **Condition 2** we always chose the first appropriate term (variable) from the sequence ST

Hence the decomposition tree definition guarantees that the decomposition process is also unique in the case of the quantifier rules $(\forall)$ and $(\exists)$

From all above, and we conclude the following

QRS Decomposition Trees

Uniqueness Theorem

For any formula $A \in \mathcal{F}$,

(i) the decomposition tree $\mathbf{T}_A$ is unique

(ii) Moreover, the following conditions hold

1. If the decomposition tree $\mathbf{T}_A$ is **finite** and all its leaves are axioms, then

$$\vdash_{QRS} A$$

2. If $\mathbf{T}_A$ is **finite** and contains a non-axiom leaf, or $\mathbf{T}_A$ is **infinite**, then

$$\nvdash_{QRS} A$$

Examples of Decomposition Trees

In all the examples below, the formulas $A(x)$, $B(x)$ represent any formulas

But as there is no indication about their particular components, they are treated as **indecomposable** formulas

For example, the decomposition tree of the formula A representing the **de Morgan Law**

$$(\neg\forall x A(x) \Rightarrow \exists x \neg A(x))$$

is constructed as follows

Examples of Decomposition Trees

$\mathbf{T}_A$

$$(\neg\forall x A(x) \Rightarrow \exists x \neg A(x))$$

$$\mid (\Rightarrow)$$

$$\neg\neg\forall x A(x), \exists x \neg A(x)$$

$$\mid (\neg\neg)$$

$$\forall x A(x), \exists x \neg A(x)$$

$$\mid (\forall)$$

$$A(x_1), \exists x \neg A(x)$$

where x_1 is a first free variable in the sequence ST such that x_1 does not appear in $\forall x A(x), \exists x \neg A(x)$

$$\mid (\exists)$$

$$A(x_1), \neg A(x_1), \exists x \neg A(x)$$

where x_1 is the first term (variables are terms) in the sequence ST such that $\neg A(x_1)$ does not appear on a tree above $A(x_1), \neg A(x_1), \exists x \neg A(x)$

Axiom

Examples of Decomposition Trees

The above tree $\mathbf{T}_A$ ended with one leaf being axiom, so it represents a **proof** in **QRS** of the **de Morgan Law**

$$(\neg\forall xA(x) \Rightarrow \exists x\neg A(x))$$

and . we have proved that

$$\vdash \ (\neg\forall xA(x) \Rightarrow \exists x\neg A(x))$$

The decomposition tree $\mathbf{T}_A$ for a formula

$$(\forall xA(x) \Rightarrow \exists xA(x))$$

is constructed as follows

Examples of Decomposition Trees

$\mathbf{T}_A$

$$(\forall x A(x) \Rightarrow \exists x A(x))$$

$$\mid (\Rightarrow)$$

$$\neg\forall x A(x), \exists x A(x)$$

$$\mid (\neg\forall)$$

$$\neg\forall x A(x), \exists x A(x)$$

$$\exists x \neg A(x), \exists x A(x)$$

$$\mid (\exists)$$

$$\neg A(t_1), \exists x A(x), \exists x \neg A(x)$$

where t_1 is the first term in the sequence ST, such that $\neg A(t_1)$ does not appear on the tree above $\neg A(t_1), \exists x A(x), \exists x \neg A(x)$

$$\mid (\exists)$$

$$\neg A(t_1), A(t_1), \exists x \neg A(x), \exists x A(x)$$

where t_1 is the first term in the sequence ST, such that $A(t_1)$ does not appear on the tree above $\neg A(t_1), A(t_1), \exists x \neg A(x), \exists x A(x)$

Axiom

Examples of Decomposition Trees

The above tree also ended with the only leaf being the axiom, hence we have **proved** that

$$\vdash \ (\forall x A(x) \Rightarrow \exists x A(x))$$

We know that the the inverse implication

$$(\exists x A(x) \Rightarrow \forall x A(x))$$

is not a predicate tautology

Let's now look at its **decomposition tree** $\mathbf{T}_A$

Examples of Decomposition Trees

$\mathbf{T}_A$

$$\exists x A(x)$$

$$\mid (\exists)$$

$$A(t_1), \exists x A(x)$$

where t_1 is the first term in the sequence ST, such that $A(t_1)$ does not appear on the tree above $A(t_1), \exists x A(x)$

$$\mid (\exists)$$

$$A(t_1), A(t_2), \exists x A(x)$$

where t_2 is the first term in the sequence ST, such that $A(t_2)$ does not appear on the tree above $A(t_1), A(t_2), \exists x A(x)$, i.e. $t_2 \neq t_1$

$$\mid (\exists)$$

$$A(t_1), A(t_2), A(t_3), \exists x A(x)$$

where t_3 is the first term in the sequence ST, such that $A(t_3)$ does not appear on the tree above $A(t_1), A(t_2), A(t_3), \exists x A(x)$, i.e. $t_3 \neq t_2 \neq t_1$

$$\mid (\exists)$$

Examples of Decomposition Trees

We continue the decomposition

$$\mid (\exists)$$

$$A(t_1), A(t_2), A(t_3), A(t_4), \exists x A(x)$$

where t_4 is the first term in the sequence ST, such that $A(t_4)$ does not appear on the tree above $A(t_1), A(t_2), A(t_3), A(t_4), \exists x A(x)$, i.e. $t_4 \neq t_3 \neq t_2 \neq t_1$

$$\mid (\exists)$$

.....

$$\mid (\exists)$$

.....

infinite branch

Obviously, the above decomposition tree is **infinite**, what proves that

$$\nvdash \exists x A(x)$$

Examples of Decomposition Trees

We construct now a **proof** in **QRS** of the quantifiers **distributivity law**

$$(\exists x(A(x) \cap B(x)) \Rightarrow (\exists xA(x) \cap \exists xB(x)))$$

and show that the proof in **QRS** of the inverse implication

$$((\exists xA(x) \cap \exists xB(x)) \Rightarrow \exists x(A(x) \cap B(x)))$$

does not exist, i.e. that

$$\nvdash \ ((\exists xA(x) \cap \exists xB(x)) \Rightarrow \exists x(A(x) \cap B(x)))$$

The decomposition tree $\mathbf{T}_A$ of the first formula is the following

Examples of Decomposition Trees

$\mathbf{T}_A$

$$(\exists x(A(x) \cap B(x)) \Rightarrow (\exists x A(x) \cap \exists x B(x)))$$

$$\mid (\Rightarrow)$$

$$\neg \exists x(A(x) \cap B(x)), (\exists x A(x) \cap \exists x B(x))$$

$$\mid (\neg \exists)$$

$$\forall x \neg (A(x) \cap B(x)), (\exists x A(x) \cap \exists x B(x))$$

$$\mid (\forall)$$

$$\neg (A(x_1) \cap B(x_1)), (\exists x A(x) \cap \exists x B(x))$$

where x_1 is a first free variable in the sequence ST such that x_1 does not appear in $\forall x \neg (A(x) \cap B(x)), (\exists x A(x) \cap \exists x B(x))$

$$\mid (\neg \cap)$$

$$\neg A(x_1), \neg B(x_1), (\exists x A(x) \cap \exists x B(x))$$

$$\bigwedge (\cap)$$

Examples of Decomposition Trees

$$\bigwedge(\cap)$$

$$\neg A(x_1), \neg B(x_1), \exists x A(x)$$

$$\mid (\exists)$$

$$\neg A(x_1), \neg B(x_1), A(t_1), \exists x A(x)$$

where t_1 is the first term in the sequence ST, such that $A(t_1)$ does not appear on the tree above $\neg A(x_1), \neg B(x_1), A(t_1), \exists x A(x)$

$$\mid (\exists)$$

$$....$$

$$\neg A(x_1), \neg B(x_1), ...A(x_1), \exists x A(x)$$

axiom

$$\neg A(x_1), \neg B(x_1), \exists x B(x)$$

$$\mid (\exists)$$

$$\neg A(x_1), \neg B(x_1), B(t_1), \exists x B(x)$$

$$\mid (\exists)$$

$$...$$

$$\mid (\exists)$$

$$\neg A(x_1), \neg B(x_1), ...B(x_1), \exists x B(x)$$

axiom

Examples of Decomposition Trees

Observe, that it is possible to choose eventually a term $t_i = x_1$, as the formula $A(x_1)$ **does not** appear on the tree above

$$\neg A(x_1), \neg B(x_1), ...A(x_1), \exists x A(x)$$

By the definition of the sequence ST, the variable x_1 is placed somewhere in it, i.e. $x_1 = t_i$, for certain $i \geq 1$

It means that after i applications of the step $(\exists)$ in the decomposition tree, we will get an axiom leaf

$$\neg A(x_1), \neg B(x_1), ...A(x_1), \exists x A(x)$$

Examples of Decomposition Trees

All leaves of the above tree $\mathbf{T}_A$ are axioms, what means that we proved

$$\vdash_{QRS} (\exists x(A(x) \cap B(x)) \Rightarrow (\exists xA(x) \cap \exists xB(x))).$$

We construct now, as the last example, a decomposition tree $\mathbf{T}_A$ of the formula

$$((\exists xA(x) \cap \exists xB(x)) \Rightarrow \exists x(A(x) \cap B(x)))$$

Examples of Decomposition Trees

$$\mathbf{T}_A$$

$$((\exists x A(x) \cap \exists x B(x)) \Rightarrow \exists x(A(x) \cap B(x)))$$

$$\mid (\Rightarrow)$$

$$\neg(\exists x A(x) \cap \exists x B(x)) \exists x(A(x) \cap B(x))$$

$$\mid (\neg\cap)$$

$$\neg\exists x A(x), \neg\exists x B(x), \exists x(A(x) \cap B(x))$$

$$\mid (\neg\exists)$$

$$\forall x \neg A(x), \neg\exists x B(x), \exists x(A(x) \cap B(x))$$

$$\mid (\forall)$$

$$\neg A(x_1), \neg\exists x B(x), \exists x(A(x) \cap B(x))$$

$$\mid (\neg\exists)$$

$$\neg A(x_1), \forall x \neg B(x), \exists x(A(x) \cap B(x))$$

$$\mid (\forall)$$

Examples of Decomposition Trees

$$\mid (\forall)$$

$$\neg A(x_1), \neg B(x_2), \exists x(A(x) \cap B(x))$$

By the reasoning similar to the reasonings in the previous examples we get that $x_1 \neq x_2$

$$\mid (\exists)$$

$$\neg A(x_1), \neg B(x_2), (A(t_1) \cap B(t_1)), \exists x(A(x) \cap B(x))$$

where t_1 is the first term in the sequence ST such that $(A(t_1) \cap B(t_1))$ does not appear on the tree above $\neg A(x_1), \neg B(x_2), (A(t_1) \cap B(t_1)), \exists x(A(x) \cap B(x))$ Observe, that it is possible that $t_1 = x_1$, as $(A(x_1) \cap B(x_1))$ does not appear on the tree above. By the definition of the sequence **??**, x_1 is placed somewhere in it, i.e. $x_1 = t_i$, for certain $i \geq 1$. For simplicity, we assume that $t_1 = x_1$ and get the sequence:

$$\neg A(x_1), \neg B(x_2), (A(x_1) \cap B(x_1)), \exists x(A(x) \cap B(x))$$

$$\bigwedge (\cap)$$

Examples of Decomposition Trees

$$\bigwedge(\cap)$$

$$\neg A(x_1), \neg B(x_2),$$
$$A(x_1), \exists x(A(x) \cap B(x))$$

Axiom

$$\neg A(x_1), \neg B(x_2),$$
$$B(x_1), \exists x(A(x) \cap B(x))$$
$$\mid (\exists)$$
$$\neg A(x_1), \neg B(x_2), B(x_1),$$
$$(A(x_2) \cap B(x_2)), \exists x(A(x) \cap B(x))$$

see COMMENT

$$\bigwedge(\cap)$$

Examples of Decomposition Trees

COMMENT: where $x_2 = t_2$ $(x_1 \neq x_2)$ is the first term in the sequence ST, such that $(A(x_2) \cap B(x_2))$ does not appear on the tree above $\neg A(x_1), \neg B(x_2), (B(x_1), (A(x_2) \cap B(x_2)), \exists x(A(x) \cap B(x))$. We assume that $t_2 = x_2$ for the reason of simplicity.

$$\bigwedge(\cap)$$

$\neg A(x_1),$	$\neg A(x_1),$
$\neg B(x_2),$	$\neg B(x_2),$
$B(x_1), A(x_2),$	$B(x_1), B(x_2),$
$\exists x(A(x) \cap B(x))$	$\exists x(A(x) \cap B(x))$
$\mid (\exists)$	*Axiom*
...	
$\mid (\exists)$	
infinite branch	

Examples of Decomposition Trees

The above decomposition tree $\mathbf{T}_A$ contains an infinite branch what means that

$$\nvdash_{QRS} ((\exists x A(x) \cap \exists x B(x)) \Rightarrow \exists x (A(x) \cap B(x)))$$

Chapter 10
Predicate Automated Proof Systems

Slides Set 1

PART 2: Proof of **QRS** Completeness

QRS Completeness

Our main goal now is to prove the Completeness Theorem for the predicate proof system **QRS**

The **proof** of the Completeness Theorem presented here is due to Rasiowa and Sikorski (1961), as is the proof system **QRS**

We adopted Rasiowa - Sikorski proof of **QRS** completeness to propositional case in chapter 6

QRS Completeness

The completeness **proofs**, in the propositional case and in predicate case, are constructive as they are based on a direct construction of a **counter model** for any unprovable formula

The construction of the **counter model** for the unprovable formula A uses in both cases the decomposition tree $\mathbf{T}_A$

Rasiowa-Sikorski type of **constructive proofs** by defining counter models determined by the decomposition trees relay heavily of the notion of **strong soundness**

QRS Semantics

Given a first order language $\mathcal{L}$

$$\mathcal{L} = \mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg\}}(\mathbf{P}, \mathbf{F}, \mathbf{C})$$

with the set *VAR* of variables and the set $\mathcal{F}$ of formulas

We define, after chapter 8 a notion of a **model** and a **counter-model** of a formula $A \in \mathcal{F}$

We establish the **semantics** for **QRS** by extending it to the the set

$$\mathcal{F}^*$$

of all finite sequences of formulas of $\mathcal{L}$

QRS Semantics

Model

A structure $\mathcal{M} = [M, I]$ is called a **model** of $A \in \mathcal{F}$

if and only if

$$(\mathcal{M}, v) \models A$$

for all assignments $v : VAR \longrightarrow M$

We denote it by

$$\mathcal{M} \models A$$

M is called the **universe** of the model, I the **interpretation**

QRS Semantics

Counter - Model

A structure $\mathcal{M} = [M, I]$ is called a **counter- model** of $A \in \mathcal{F}$ if and only if **there is** $v : VAR \longrightarrow M$, such that

$$(\mathcal{M}, v) \not\models A$$

We denote it by

$$\mathcal{M} \not\models A$$

QRS Semantics

Tautology

A formula $A \in \mathcal{F}$ is called a **predicate tautology** and denoted by $\models A$ if and only if

all structures $\mathcal{M} = [M, I]$ are **models** of A, i.e.

$$\models A \quad \text{if and only if} \quad \mathcal{M} \models A$$

for all structures $\mathcal{M} = [M, I]$ for $\mathcal{L}$

QRS Semantics

For any sequence $\Gamma \in \mathcal{F}^*$, by δ_Γ we understand any **disjunction** of all formulas of Γ

A structure $\mathcal{M} = [M, I]$ is called a **model** of a sequence $\Gamma \in \mathcal{F}^*$ and denoted by

$$\mathcal{M} \models \Gamma$$

if and only if $\mathcal{M} \models \delta_\Gamma$

The sequence $\Gamma \in \mathcal{F}^*$ is a **predicate tautology** if and only if the formula δ_Γ is a predicate tautology, i.e.

$$\models \Gamma \text{ if and only if } \models \delta_\Gamma$$

Strong Soundnesss

Our **goal** now is to prove the Completeness Theorem for **QRS**

The **correctness** of the Rasiowa-Sikorski **constructive proof** depends on the strong soundness of the rules of inference of **QRS**

We define it (in general case) as follows

Strong Soundnesss

Strongly Sound Rules

Given a predicate language proof system

$$S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$$

An inference rule $r \in \mathcal{R}$ of the form

$$(r) \quad \frac{P_1 \ ; \ P_2 \ ; \ .... \ ; \ P_m}{C}$$

is **strongly sound** if the following condition holds for any structure $\mathcal{M} = [M, I]$ for $\mathcal{L}$

$$\mathcal{M} \models \{P_1, P_2, .P_m\} \text{ if and only if } \mathcal{M} \models C$$

Strong Soundnesss

A predicate language proof system $S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$ is **strongly sound** if and only if all logical axioms LA are tautologies and all its rules of inference $r \in \mathcal{R}$ are strongly sound

Strong Soundness Theorem
The proof system **QRS** is strongly sound

Proof
We have already proved in chapter 6 strong soundness of the propositional rules. The quantifiers rules are strongly sound by straightforward verification and is left as an exercise

Soundnesss Theorem

The strong soundness property is stronger then soundness property, hence also the following holds

QRS Soundness Theorem

For any $\Gamma \in \mathcal{F}^*$,

$$\text{if} \quad \vdash_{QRS} \ \Gamma, \quad \text{then} \quad \models \ \Gamma$$

In particular, for any formula $A \in \mathcal{F}$,

$$\text{if} \quad \vdash_{QRS} A, \quad \text{then} \models \ A$$

Proof of Completeness Theorem

Completeness Theorem

For any $\Gamma \in \mathcal{F}^*$,

$$\vdash_{QRS} \Gamma \quad \text{if and only if} \quad \models \Gamma$$

In particular, for any formula $A \in \mathcal{F}$,

$$\vdash_{QRS} A \quad \text{if and only if} \quad \models A$$

Proof We prove the completeness part. We need to prove the formula A case only because the case of a sequence Γ can be reduced to the formula case of δ_Γ. I.e. we prove the implication:

$$\text{if } \models A, \quad \text{then } \vdash_{QRS} A$$

Proof of Completeness Theorem

We do it, as in the propositional case, by proving the opposite implication

$$\text{if} \quad \nvdash_{QRS} \; A \quad \text{then} \quad \not\models \; A$$

This means that we want prove that for any formula A, **unprovability** of A in **QRS** allows us to define its **counter-model**.

The counter- model is determined, as in the propositional case, by the decomposition tree $\mathbf{T}_A$

We have proved the following

Tree Theorem

Each formula A, generates its unique decomposition tree $\mathbf{T}_A$ and A has a proof only if this tree is finite and all its end sequences (leaves) are axioms

Proof of Completeness Theorem

The **Tree Theorem** says says that we have two cases to consider:

(C1) the tree $\mathbf{T}_A$ is **finite** and contains a leaf which is not axiom, or

(C2) the tree $\mathbf{T}_A$ is **infinite**

We will show how to construct a counter- model for A in both cases:

a counter- model determined by a non-axiom leaf of the decomposition tree $\mathbf{T}_A$,

or a counter- model determined by an infinite branch of $\mathbf{T}_A$

Proof of Completeness Theorem

Proof in case **(C1)**

The tree $\mathbf{T}_A$ is **finite** and contains a non- axiom leaf

Before describing a general method of constructing the counter-model determined by the decomposition tree $\mathbf{T}_A$ we describe it, as an example, for a case of a general formula

$$(\exists x A(x) \Rightarrow \forall x A(x)),$$

and its **particular case**

$$(\exists x(P(x) \cap R(x, y)) \Rightarrow \forall x(P(x) \cap R(x, y))),$$

where P, R are one and two argument predicate symbols, respectively

Proof of Completeness Theorem

First we build its decomposition tree:

$$\mathbf{T}_A$$

$$(\exists x(P(x) \cap R(x,y)) \Rightarrow \forall x(P(x) \cap R(x,y)))$$

$$\mid (\Rightarrow)$$

$$\neg\exists x(P(x) \cap R(x,y)), \forall x(P(x) \cap R(x,y))$$

$$\mid (\neg\exists)$$

$$\forall x \neg(P(x) \cap R(x,y)), \forall x(P(x) \cap R(x,y))$$

$$\mid (\forall)$$

$$\neg(P(x_1) \cap R(x_1,y)), \forall x(P(x) \cap R(x,y))$$

where x_1 is a first free variable in the sequence of term ST such that x_1 does not appear in $\forall x \neg(P(x) \cap R(x,y)), \forall x(P(x) \cap R(x,y))$

$$\mid (\neg\cap)$$

$$\neg P(x_1), \neg R(x_1,y), \forall x(P(x) \cap R(x,y))$$

$$\mid (\forall)$$

Proof of Completeness Theorem

$$\mid (\forall)$$

$$\neg P(x_1), \neg R(x_1, y), (P(x_2) \cap R(x_2, y))$$

where x_2 is a first free variable in the sequence of term ST such that x_2 does not appear in $\neg P(x_1), \neg R(x_1, y), \forall x(P(x) \cap R(x, y))$, the sequence ST is one-to- one, hence $x_1 \neq x_2$

$$\bigwedge (\cap)$$

$$\neg P(x_1), \neg R(x_1, y), P(x_2)$$

$x_1 \neq x_2$, Non-axiom

$$\neg P(x_1), \neg R(x_1, y), R(x_2, y)$$

$x_1 \neq x_2$, Non-axiom

Proof of Completeness Theorem

There are two non-axiom leaves

In order to define a counter-model determined by the tree $\mathbf{T}_A$ we need to chose only one of them

Let's choose the leaf

$$L_A = \neg P(x_1), \neg R(x_1, y), P(x_2)$$

We use the **non-axiom leaf** L_A to define a structure $\mathcal{M} = [M, I]$ and an assignment v, such that

$$(\mathcal{M}, v) \not\models A$$

Such defined $\mathcal{M}$ is called a **counter - model** determined by the tree $\mathbf{T}_A$

Proof of Completeness Theorem

We take a the **universe** of $\mathcal{M}$ the set $\mathbf{T}$ of all terms of the language $\mathcal{L}$, i.e. we put $M = \mathbf{T}$.

We define the **interpretation** I as follows.

For any **predicate** symbol $Q \in \mathbf{P}, \#Q = n$ we put that

$Q_I(t_1, \dots t_n)$ is **true** (holds) for terms $t_1, \dots t_n$

if and only if

the negation $\neg Q_I(t_1, \dots t_n)$ of the formula $Q(t_1, \dots t_n)$ **appears** on the leaf L_A

and $Q_I(t_1, \dots t_n)$ is **false** (does not hold) for terms $t_1, \dots t_n$, otherwise

For any **functional** symbol $f \in \mathbf{F}, \#f = n$ we put

$$f_I(t_1, \dots t_n) = f(t_1, \dots t_n)$$

Proof of Completeness Theorem

It is easy to see that in particular case of our non-axiom leaf

$$L_A = \neg P(x_1),\ \neg R(x_1, y),\ P(x_2)$$

$P_I(x_1)$ is **true** (holds) for x_1, and **not true** for x_2

$R_I(x_1, y)$ is **true** (holds) for x_1 and for any $y \in VAR$

Proof of Completeness Theorem

We define the assignment $v : VAR \longrightarrow T$ as **identity**,

i.e., we put $v(x) = x$ for any $x \in VAR$

Obviously, for such defined structure $[M, I]$ and the assignment v we have that

$$([\mathbf{T}, I], v) \models P(x_1), \quad ([\mathbf{T}, I], v) \models R(x_1, y), \quad ([\mathbf{T}, I], v) \not\models P(x_2)$$

We hence obtain that

$$([\mathbf{T}, I],\ v) \not\models \neg P(x_1), \neg R(x_1, y), P(x_2)$$

This proves that such defined structure $[\mathbf{T}, I]$ is a **counter model** for a non-axiom leaf L_A and by the **Strong Soundness** we proved that

$$\not\models (\exists x(P(x) \cap R(x, y)) \Rightarrow \forall x(P(x) \cap R(x, y)))$$

C1: Proof of Completeness Theorem

C1: General Method

Let A be any formula such that

$$\nvdash_{QRS} \ A$$

Let $\mathbf{T}_A$ be a decomposition tree of A

By the fact that $\nvdash_{QRS}$ and **C1**, the tree $\mathbf{T}_A$ is **finite** and has a non axiom leaf

$$L_A \subseteq LT^*$$

By definition, the leaf L_A contains only atomic formulas and negations of atomic formulas

C1: Counter Model Definition

We use the **non-axiom leaf** L_A to define a structure $\mathcal{M} = [M, I]$, an assignment $v : VAR \longrightarrow M$, such that

$$(\mathcal{M}, v) \not\models A$$

Such defined structure $\mathcal{M}$ is called a **counter - model** determined by the tree $\mathbf{T}_A$

C1: Counter Model Definition

Structure $\mathcal{M}$ **Definition**

Given a formula A and a **non-axiom** leaf L_A

We define a structure

$$\mathcal{M} = [M, I]$$

and an assignment $v : VAR \longrightarrow M$ as follows

1. We take a the universe of $\mathcal{M}$ the set $\mathbf{T}$ of all **terms** of the language $\mathcal{L}$, i.e. we put

$$M = \mathbf{T}$$

C1: Counter Model Definition

2. For any predicate symbol $Q \in \mathbf{P}, \#Q = n$,

$$Q_I \subseteq \mathbf{T}^n$$

is such that $Q_I(t_1, \dots t_n)$ **holds** (is true) for terms $t_1, \dots t_n$

if and only if

the **negation** $\neg Q(t_1, \dots t_n)$ of the formula $Q(t_1, \dots t_n)$ appears on the leaf L_A and

$Q_I(t_1, \dots t_n)$ **does not hold** (is false) for terms $t_1, \dots, t_n$ otherwise

C1: Counter Model Definition

3. For any constant $c \in \mathbf{C}$, we put $c_I = c$

For any variable x, we put $x_I = x$

For any functional symbol $f \in \mathbf{F}$, $\# f = n$

$$f_I : \mathbf{T}^n \longrightarrow \mathbf{T}$$

is **identity** function, i.e. we put

$$f_I(t_1, \ldots t_n) = f(t_1, \ldots t_n)$$

for all $t_1, \ldots t_n \in \mathbf{T}$

4. We define the assignment $v : VAR \longrightarrow \mathbf{T}$ as identity, i.e. we put for all $x \in VAR$

$$v(x) = x$$

C1: Counter Model Definition

Obviously, for such defined structure $[\mathbf{T}, I]$ and the assignment v we have that

$$([\mathbf{T}, I], \ v) \not\models \ P \text{ if formula } P \text{ appears in } L_A,$$

$$([\mathbf{T}, I], \ v) \models \ P \text{ if formula } \neg P \text{ appears in } L_A$$

This proves that the structure $\mathcal{M} = [\mathbf{T}, I]$ and assignment v are such that

$$([\mathbf{T}, I], v) \not\models \ L_A$$

C1: Counter Model Definition

By the **Strong Soundness Theorem** we have that

$$(([\mathbf{T}, I], v) \not\models A$$

This proves $\mathcal{M} \not\models A$ and we proved that

$$\not\models A$$

This **ends** the proof of the case **C1**

C2: Counter Model Definition

Proof of case **C2**: $\mathbf{T}_A$ is **infinite**

The case of the **infinite tree** is similar to the **C1** case, even if a little bit more complicated

Observe that the rule $(\exists)$ is the **only** rule of inference (decomposition) which can "produces" an infinite branch

We first show how to construct the **counter-model** in the case of the simplest application of this rule, i.e. in the case of the atomic formula

$$\exists x P(x)$$

for P one argument relational symbol. All other cases are. similar to this one

C2: Particular Case n

The **infinite** branch $\mathcal{B}_A$ in the following

$$\mathcal{B}_A$$

$$\exists x P(x)$$

$$\mid (\exists)$$

$$P(t_1), \exists x P(x)$$

where t_1 is the first term in the sequence of terms, such that $P(t_1)$ does not appear on the tree above $P(t_1), \exists x P(x)$

$$\mid (\exists)$$

$$P(t_1), P(t_2), \exists x P(x)$$

where t_2 is the first term in the sequence of terms, such that $P(t_2)$ does not appear on the tree above $P(t_1), P(t_2), \exists x P(x)$, i.e. $t_2 \neq t_1$

$$\mid (\exists)$$

C2: Particular Case

$$| \, (\exists)$$

$$P(t_1), P(t_2), P(t_3), \exists x P(x)$$

where t_3 is the first term in the sequence of terms, such that $P(t_3)$ does not appear on the tree above $P(t_1), P(_2), P(t_3), \exists x P(x)$, i.e. $t_3 \neq t_2 \neq t_1$

$$| \, (\exists)$$

$$P(t_1), P(t_2), P(t_3), P(t_4), \exists x P(x)$$

$$| \, (\exists)$$

.....

$$| \, (\exists)$$

.....

The infinite branch $\mathcal{B}_A$, written from the top, in oder of appearance of formulas is

$$\mathcal{B}_A = \{\exists x P(x), \; P(t_1), \; A(t_2), \; P(t_2), \; P(t_4), ...\}$$

where $t_1, t_2, ...$ is a one - to one sequence of **all terms**

C2: Particular Case n

The **infinite** branch

$$\mathcal{B}_A = \{\exists x P(x),\ P(t_1),\ A(t_2),\ P(t_2),\ P(t_4), ...\}$$

contains with the formula $\exists x P(x)$ all its instances $P(t)$, for all terms $t \in \mathbf{T}$

We define the structure $\mathcal{M} = [M, I]$ and the assignment v as we did previously, i.e.

we take as the universe M the set $\mathbf{T}$ of all terms, and define P_I as follows:

$P_I(t)$ **holds** if $\neg P(t) \in \mathcal{B}_A$, and

$P_I(t)$ **does not hold** if $P(t) \in \mathcal{B}_A$

C2: Particular Case

For any constant $c \in \mathbf{C}$, we put $c_I = c$, for any variable x, we put $x_I = x$

For any functional symbol $f \in \mathbf{F}$, $\# f = n$

$$f_I : \mathbf{T}^n \longrightarrow \mathbf{T}$$

is **identity** function, i.e. we put

$$f_I(t_1, \ldots t_n) = f(t_1, \ldots t_n)$$

for all $t_1, \ldots t_n \in \mathbf{T}$

C2: Particular Case

We define the assignment $v : VAR \longrightarrow \mathbf{T}$ as identity, i.e. we put for all $x \in VAR$

$$v(x) = x$$

It is easy to see that for any formula $P(t) \in \mathcal{B}$,

$$([T, I], v) \not\models P(t)$$

But the $P(t) \in \mathcal{B}$ are **all instances** of the formula $\exists x P(x)$, hence

$$([T, I], v) \not\models \exists x P(x)$$

and we proved

$$\not\models \exists x P(x)$$

C2: General Method

C2: General Method

Let A be any formula such that

$$\nvdash_{QRS} \; A$$

Let $\mathcal{T}_A$ be an **infinite** decomposition tree of the formula A

Let $\mathcal{B}_A$ be the **infinite branch** of $\mathbf{T}_A$, written from the top, in order of appearance of sequences $\Gamma \in \mathcal{F}^*$ on it, where $\Gamma_0 = A$, i.e.

$$\mathcal{B}_A = \{\Gamma_0, \; \Gamma_1, \; \Gamma_2, \; \ldots \; \Gamma_i, \; \Gamma_{i+1}, \; \ldots\}$$

C2: General Method

Given the infinite branch

$$\mathcal{B}_A = \{\Gamma_0,\ \ \Gamma_1,\ \ \Gamma_2,\ \ \ldots\ \ \Gamma_i,\ \ \Gamma_{i+1},\ \ \ldots\}$$

We define a set

$$L\mathcal{F} \subseteq \mathcal{F}$$

of all **indecomposable** formulas appearing in at least one sequence $\Gamma_i,\ i \leq j$, i.e. we put

$$L\mathcal{F} = \{B \in LT : \text{ there is } \Gamma_i \in \mathcal{B}_A, \text{ such that } B \text{ iappiears } \Gamma_i\}$$

C2: General Method

Note, that the following holds

(1) If $i \leq i'$ and an **indecomposable** formula appears in Γ_i, then it also appears in $\Gamma_{i'}$

(2) Since **none** of Γ_i is an axiom, for every atomic formula $P \in A\mathcal{F}$, at **most one** of the formulas P and $\neg P$ is in $L\mathcal{F}$

Counter Model Definition

Let $\mathbf{T}$ be the set of all terms. We define the structure $\mathcal{M} = [\mathbf{T}, I]$, the interpretation $\mathbf{I}$ of constants and functional symbols, and the assignment v in the set $\mathbf{T}$, as in previous cases

We define the interpretation $\mathbf{I}$ of predicates $Q \in \mathbf{P}$ as follows

For any predicate symbol $Q \in \mathbf{P}$, $\# Q = n$, we put

(1) $Q_I(t_1, \dots t_n)$ **does not hold** (is false) for terms $t_1, \dots t_n$ if and only if

$$Q_I(t_1, \dots t_n) \in L\mathcal{F}$$

(2) $Q_I(t_1, \dots t_n)$ **does holds** (is true) for terms $t_1, \dots t_n$ if and only if

$$[Q_I(t_1, \dots t_n) \notin L\mathcal{F}$$

Counter Model Definition

Directly from the definition we we have that $\mathcal{M} \not\models \ L\mathcal{F}$

Our goal now is to prove that

$$\mathcal{M} \not\models \ A$$

For this purpose we first introduce, for any formula $A \in \mathcal{F}$, an inductive definition of the **order** $ordA$ of the formula A

(1) If $A \in A\mathcal{F}$, then $ord\ A = 1$

(2) If $ordA = n$, then $ord\neg A = n + 1$

(3) If $ordA \leq n$ and $ordB \leq n$, then
$ord(A \cup B) = ord(A \cap B) = ord(A \Rightarrow B) = n + 1$

(4) If $ordA(x) = n$, then $ord\exists xA(x) = ord\forall xA(x) = n + 1$

Proof of Completeness Theorem

We conduct the proof of $\mathcal{M} \not\models A$ by contradiction.

Assume that

$$\mathcal{M} \models A$$

Consider now a set $M\mathcal{F}$ of all formulas B appearing in one of the sequences Γ_i of the branch $\mathcal{B}_A$, such that

$$\mathcal{M} \models B$$

We write the the set $M\mathcal{F}$ formally as follows

$$M\mathcal{F} = \{B \in \mathcal{F} : \text{for some } \Gamma_i \in \mathcal{B}_A,\ B \text{ is in } \Gamma_i \text{ and } \mathcal{M} \models B\}$$

Proof of Completeness Theorem

Observe that the formula A is in $M\mathcal{F}$ so

$$M\mathcal{F} \neq \emptyset$$

Let B' be a formula in $M\mathcal{F}$ such that

$$ordB' \leq ordB \quad \text{for every} \quad B \in M\mathcal{F}$$

There exists $\Gamma_i \in\in \mathcal{B}_A$ that is of the form Γ', B', Δ with an **indecomposable** Γ'

We have that B' **can not** be of the form

$$(*) \quad \neg\exists x A(x) \quad \text{or} \quad \neg\forall x A(x)$$

for if B' of the $(*)$ form **is** in $M\mathcal{F}$, then also formula $\forall x \neg A(x)$ or $\exists x \neg A(x)$ is in $M\mathcal{F}$ and the **orders** of the two formulas are equal

Proof of Completeness Theorem

We carry the same order argument and show that B' **can not** be of the form

$$(**) \quad (A \cup B), \ \neg(A \cup B), \ (A \cap B), \ \neg(A \cap B),$$

$$(A \Rightarrow B), \ \neg(A \Rightarrow B), \ \neg\neg A, \ \forall x A(x)$$

The formula B' **can not** be of the form

$$(***) \quad \exists x B(x)$$

since then there exists term t and j such that $i \leq j$, and $B'(t)$ **appears** in Γ_j and the formula $B(t)$ is such that

$$\mathcal{M} \models B$$

Proof of Completeness Theorem

Thus $B(t) \in M\mathcal{F}$ and $ordB(t) < ordB'$

This **contradicts** the definition of B'

Since B' **is not** of the forms $(*)$, $(**)$, $(***)$, B' is **indecomposable**. Thus $B' \in L\mathcal{F}$ and consequently

$$\mathcal{M} \not\models B'$$

On the other hand B' is in the set $M\mathcal{F}$ and hence is one of the formulas satisfying

$$\mathcal{M} \models B'$$

This **contradiction** proves that $\mathcal{M} \not\models A$ and hence we proved that

$$\not\models A$$

This **ends** the proof of the Completeness Theorem for **QRS**