CSE371 Practice Midterm SOLUTION Fall 2009

QUESTION 1

(a) Describe shortly a difference between *logical* and *semantical* paradoxes.

Solution: Logical paradoxes deal with the notion of a set.

Semantical paradoxes deal with the notion of truth, or provability. They are caused by a collision between the theory and meta-theory, that is, by inclusion of meta-theoretical statements in the theory.

(b) Give an example (by name) of a logical paradox.

Russell, 1902 Consider the set A of all those sets X such that X is not a member of X. Clearly, by definition, A is a member of A if and only if A is not a member of A. So, if A is a member of A, the A is also not a member of A; and if A is not a member of A, then A is a member of A. In any case, A is a member of A and A is not a member of A.

Cantor, Burali-Forti antinomies. See Chapter 1.

(c) Describe shortly a difference between classical and intuitionistic logic.

Solution: Intuitionistic logic does not accept the universality of certain basic classical logical laws, such as the law of *excluded middle*: $(A \cup \neg A)$.

The classical and intuitionistic logic **also differ** on the interpretation of the meaning of the word *exists* and in the interpretation of logical implication, i.e. the truthfulness of the sentences of the form *if* A *then* B is decided differently in both logics.

QUESTION 2

(1) For the sentence below write its corresponding formula A. Explain your solution.

If from the fact that all sides of a triangle ABC are equal we can deduce that all angles of the triangle ABC are equal and all angles of the triangle ABC are not equal, then all sides of a triangle ABC are equal.

Solution: We denote *"all sides of a triangle ABC are equal"* $= a$, *"all angles of the triangle ABC are equal"* $=b$.

The formula A is

$$((a \Rightarrow (b \cap \neg b)) \Rightarrow a).$$

(2) Define a formal language to which the formula A belongs.

Solution: The language is $\mathcal{L}_{\{\cap, \Rightarrow, \neg\}}$, or any $\mathcal{L}_{CON}$ for the set of connectives CON such that $\{\cap, \Rightarrow, \neg\} \subseteq CON$.

(3) Determine the degree of A and write down all its sub-formulas of the degree 2.

Solution: $degree(A) = 4$, there is only one sub-formula of the degree 2: $(b \cap \neg b)$.

(4) Determine the following: $A \in \mathbf{T}, A \in \mathbf{C}$.

You can use the shorthand notation for your work, but we write here solutions using proper definitions.

Solution: (this is the most formal solution!)

Part one: for $A \in \mathbf{T}$.

We use the following definition.

$$\models A \;\; if \;\; only \;\; if \;\; v^*(A) = T$$

for all $v : VAR \rightarrow \{T, F\}$ and v^* is defined by classical semantics.

Proof by contradiction. Assume $A \notin \mathbf{T}$.I.e. there is v, such that $v^*(A) = F$.
We evaluate $v^*((a \Rightarrow (b \cap \neg b)) \Rightarrow a) = ((v^*(a) \Rightarrow (v^*(b) \cap \neg v^*(b))) \Rightarrow v^*(a)) = ((v(a) \Rightarrow (v(b) \cap \neg v(b))) \Rightarrow v(a)) = F$ if and only if $(v(a) \Rightarrow (v(b) \cap \neg v(b))) = T$ and $v(a) = F$ if and only if $(F \Rightarrow (v(b) \cap \neg v(b))) = T$. This is possible when $v(b) = T$ or $v(b) = F$.

This proves that $A \notin \mathbf{T}$ and there are two counter-models (essential) for A:
(1) v such that $v(a) = F, v(b) = F$ and
(2)v such that $v(a) = F, v(b) = T$.

Solution: (this is the most formal solution)

Part two: for $A \in \mathbf{C}$.

We use the following definition.

$$A \in \mathbf{C} \;\; if \;\; and \;\; only \;\; if \;\; v^*(A) = F$$

for all $v : VAR \rightarrow \{T, F\}$ and v^* is defined by classical semantics.

Any truth assignment v such that $v(a) = T$ is a model for A, hence $A \notin \mathbf{C}$.

(5) Determine the following: $A \in \text{Ł}\mathbf{T}, A \in \mathbf{HT}$. Use a shorthand notation.

Solution 1: Any counter-model v from point **(4)** is also a **Ł** or **H** counter-model. I.e. let $v : VAR \rightarrow \{T, \perp, F\}$ be such $v(a) = F, v(b) = F$, we get $v^*((a \Rightarrow (b \cap \neg b)) \Rightarrow a) = F$, where v^* is defined by **Ł** or **H**. Hence $A \notin \text{Ł}\mathbf{T}, A \notin \mathbf{HT}$.

Solution 2: We have proved that

$\text{Ł}\mathbf{T} \subseteq \mathbf{T}$ and $\mathbf{HT} \subseteq \mathbf{T}$
$A \notin \mathbf{T}$, hence $A \notin \text{Ł}\mathbf{T}, A \notin \mathbf{HT}$.

QUESTION 3 Write the formula A from Question 2 as a formula of the language $\mathcal{L}_{\{\neg, \cup\}}$, i.e. as a formula B of $\mathcal{L}_{\{\neg, \cup\}}$, such that $A \equiv B$. Write down all logical equivalences you need while solving this problem.

Solution: We use the following logical equivalences.

1. $(A \cap B) \equiv \neg(\neg A \cup \neg B)$
2. $(A \Rightarrow B) \equiv (\neg A \cup B)$
3. $\neg(A \Rightarrow B) \equiv (A \cap \neg B)$
4. $\neg(A \cap B) \equiv (\neg A \cup \neg B)$

5. $\neg\neg A \equiv A$

We evaluate B as follows.

$((a \Rightarrow (b \cap \neg b)) \Rightarrow a) \equiv^2 \neg((a \Rightarrow (b \cap \neg b)) \cup a) \equiv^3 ((a \cap \neg(b \cap \neg b)) \cup a) \equiv^4 ((a \cap (\neg b \cup \neg\neg b)) \cup a) \equiv^4 (a \cap (\neg b \cup b)) \cup a) \equiv^4 (\neg(\neg a \cup \neg(\neg b \cup \neg\neg b)) \cup a)$.

Hence $\quad B = (\neg(\neg a \cup \neg(\neg b \cup \neg\neg b)) \cup a)$.

QUESTION 4

S is the following proof system:

$$S = (\ \mathcal{L}_{\{\Rightarrow, \cup, \neg\}}, \mathcal{F}, \quad A1, \quad (r1), \ (r2)\ )$$

Axiom

A1 $\quad (A \Rightarrow (A \cup B))$,

Rules of inference:

$$(r1)\ \frac{A\ ;B}{(A \cup \neg B)}, \qquad (r2)\ \frac{A\ ;(A \cup B)}{B}.$$

1. Verify whether S is sound/not sound under classical semantics.

Solution The system is not sound. Take any v, such that it evaluates $A = T$ and $B = F$. The premiss $(A \cup B$ of the rule $(r2)$ is T and the conclusion i F.

2. Find a formal proof of $\quad \neg(A \Rightarrow (A \cup B)) \quad$ in S, ie. show that

$$\vdash_S \neg(A \Rightarrow (A \cup B)).$$

Solution The proof is as follows.

B_1 $(A \Rightarrow (A \cup B))$,

B_2 $(A \Rightarrow (A \cup B))$,

B_3 $((A \Rightarrow (A \cup B)) \cup \neg(A \Rightarrow (A \cup B)))$,

B_4 $\neg(A \Rightarrow (A \cup B))$.

3. Does above point **2.** prove that $\models \neg(A \Rightarrow (A \cup B))$?

Solution No, the proof used rule $(r2)$ that is not sound.

QUESTION 5 THIS IS AN EXTRA QUESTION - TO STUDY FOR REAL MIDTERM.

H is the following proof system:

$$H = (\ \mathcal{L}_{\{\Rightarrow, \neg\}}, \quad A1, A2, A3, \quad MP\ )$$

A1 $\quad (A \Rightarrow (B \Rightarrow A))$,

A2 $\quad ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$

MP Rule of inference:

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

(1) Prove that H is SOUND under classical semantics.

Solution: Soundness Theorem holds because all axioms of H are tautologies and MP leads from tautologies to a tautology.

(2) Prove that H is not sound under **K** semantics.

Solution: Axiom $A1$ is not **K** tautology. Any v for which $A = B = \perp$ evaluates $A1 = \perp$.

QUESTION 6 THIS IS AN EXTRA QUESTION - TO STUDY FOR REAL MIDTERM.

Give an example of a sound (classical semantics) proof system S based on $\mathcal{L}_{\{\Rightarrow,\neg\}}$ with 2 axioms different that axioms from QUESTIONS 5 and one two premisses rule of inference that is not MP.

Solution 1:

$$S = (\mathcal{L}_{\{\cup,\neg\}},\ \ \mathcal{F},\ \ AX = \{(A \cup \neg A)\},\ \ \ \ (r)\ \frac{B\ ;A}{(\neg B \cup (A \cup \neg A))})$$

Soundness of S:

Axiom $(A \cup \neg A)$ is a basic tautology, so it is sound.

Rule of inference (r) is also sound. For let $B = T, A = T$ (both premisses true), we evaluate logical value of the conclusion as follows.

$(\neg B \cup (A \cup \neg A)) = (F \cup T) = T$.

Solution 2: Any other correct system you invent.