

## CSE371 Midterm Solutions Fall 2009

### QUESTION 1

(a) Give example (by name ) of three non-classical logics.

**Solution:** Intuitionistic Logic, Modal Logic S4, S5, and any of CS logics listed below.

(b) Give example (by name ) of two logics developed by computer scientists.

**Solution:** *Dynamic logic* (Harel 1979) which was created to facilitate the statement and proof of properties of programs.

*Temporal Logics* which were created for the specification and verification of concurrent programs Harel, Parikh, 1979, 1983 and for a specification of hardware circuits Halpern, Manna and Maszkowski, (1983).

*Fuzzy logic, Many valued logics* that were created and developed to describe reasoning with incomplete information.

*Non-monotonic logics* were created by Mc Carthy (1985) and has been shown to be important in other areas. There are applications to logic programming, to planning and reasoning about action, and to automated diagnosis.

### QUESTION 2 Let $A$ be a formula

$$A = (((a \cap \neg c) \Rightarrow \neg b) \cup a) \Rightarrow (c \cup b).$$

(1) Define a formal language to which the formula  $A$  belongs.

**Solution:** The language is  $\mathcal{L}_{\{\neg, \cap, \Rightarrow\}}$ .

(2) Determine the degree of  $A$  and write down all its sub-formulas of the degree 2.

**Solution:** The degree of  $A$  is 7.

There is only one sub-formula of the degree 2:  $(a \cap \neg c)$ .

(3) Determine the following:  $A \in \mathbf{T}, A \in \mathbf{C}$ . You can use the shorthand notation.

**Solution** of the case  $A \in \mathbf{T}$ .

Assume  $(((a \cap \neg c) \Rightarrow \neg b) \cup a) \Rightarrow (c \cup b) = F$ . This is possible if and only if  $(((a \cap \neg c) \Rightarrow \neg b) \cup a) = T$  and  $(c \cup b) = F$ . This gives as that  $c = F, b = F$ . We evaluate  $(((a \cap \neg F) \Rightarrow \neg F) \cup a) = T$ . This is possible for  $a = T$ .

Any truth assignment such that  $a = T, b = F, c = F$  is a counter-model for  $A$ , hence  $A \notin \mathbf{T}$ .

**Solution** of the case  $A \in \mathbf{C}$ .

Any truth assignment such that  $a = T, b = T, c = F$  is a model for  $A$ , hence  $A \notin \mathbf{C}$ . This is not the only model.

(4) Determine the following: (use a shorthand notation)

(a)  $A \in \mathbf{LT}$ .

**Solution 1:** We have proved that  $\mathbf{LT} \subseteq \mathbf{T}$  and  $A \notin \mathbf{T}$ , hence  $A \notin \mathbf{LT}$ .

**Solution 2:** Any truth assignment such that  $a = T, b = F, c = F$  is a counter-model for  $A$ , hence  $A \notin \mathbf{LT}$ .

**L semantics** is defined as follows.

**L Negation**

$\neg$	F	$\perp$	T
	T	$\perp$	F

**L Disjunction**

$\cup$	F	$\perp$	T
F	F	$\perp$	T
$\perp$	$\perp$	$\perp$	T
T	T	T	T

**L Conjunction**

$\cap$	F	$\perp$	T
F	F	F	F
$\perp$	F	$\perp$	$\perp$
T	F	$\perp$	T

**L-Implication**

$\Rightarrow$	F	$\perp$	T
F	T	T	T
$\perp$	$\perp$	T	T
T	F	$\perp$	T

We define a 4 valued  $\mathbf{L}_4$  logic semantics as follows. The language is

$$\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}.$$

We define the logical connectives  $\neg, \Rightarrow, \cup, \cap$  of  $\mathbf{L}_4$  as the following operations in the set  $\{F, \perp_1, \perp_2, T\}$ , where  $\{F < \perp_1 < \perp_2 < T\}$ .

**Negation  $\neg$**

$$\neg : \{F, \perp_1, \perp_2, T\} \longrightarrow \{F, \perp_1, \perp_2, T\},$$

such that

$$\neg \perp_1 = \perp_1, \quad \neg \perp_2 = \perp_2, \quad \neg F = T, \quad \neg T = F.$$

**Conjunction  $\cap$**

$$\cap : \{F, \perp_1, \perp_2, T\} \times \{F, \perp_1, \perp_2, T\} \longrightarrow \{F, \perp_1, \perp_2, T\}$$

such that for any  $a, b \in \{F, \perp_1, \perp_2, T\}$ ,

$$a \cap b = \min\{a, b\}.$$

**Disjunction  $\cup$**

$$\cup : \{F, \perp_1, \perp_2, T\} \times \{F, \perp_1, \perp_2, T\} \longrightarrow \{F, \perp_1, \perp_2, T\}$$

such that for any  $a, b \in \{F, \perp_1, \perp_2, T\}$ ,

$$a \cup b = \max\{a, b\}.$$

**Implication**  $\Rightarrow$

$$\Rightarrow: \{F, \perp_1, \perp_2, T\} \times \{F, \perp_1, \perp_2, T\} \longrightarrow \{F, \perp_1, \perp_2, T\},$$

such that for any  $a, b \in \{F, \perp_1, \perp_2, T\}$ ,

$$a \Rightarrow b = \begin{cases} \neg a \cup b & \text{if } a > b \\ T & \text{otherwise} \end{cases}$$

## QUESTION 2

**Part 1** Write Truth Tables for  $\mathbf{L}_4$

**Solution :**

**$\mathbf{L}_4$  Negation**

$\neg$	F	$\perp_1$	$\perp_2$	T
	T	$\perp_1$	$\perp_2$	F

**$\mathbf{L}_4$  Conjunction**

$\cap$	F	$\perp_1$	$\perp_2$	T
F	F	F	F	F
$\perp_1$	F	$\perp_1$	$\perp_1$	$\perp_1$
$\perp_2$	F	$\perp_1$	$\perp_2$	$\perp_2$
T	F	$\perp_1$	$\perp_2$	T

**$\mathbf{L}_4$  Disjunction**

$\cup$	F	$\perp_1$	$\perp_2$	T
F	F	$\perp_1$	$\perp_2$	T
$\perp_1$	$\perp_1$	$\perp_1$	$\perp_2$	T
$\perp_2$	$\perp_2$	$\perp_2$	$\perp_2$	T
T	T	T	T	T

**$\mathbf{L}_4$ -Implication**

$\Rightarrow$	F	$\perp_1$	$\perp_2$	T
F	T	T	T	T
$\perp_1$	$\perp_1$	T	T	T
$\perp_2$	$\perp_2$	$\perp_2$	T	T
T	F	$\perp_1$	$\perp_2$	T

**Part 2** Verify whether

$$\models_{\mathbf{L}_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b))$$

**Solution :** Let  $v$  be a truth assignment such that  $v(a) = v(b) = \perp_1$ .

We evaluate  $v^*((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = ((\perp_1 \Rightarrow \perp_1) \Rightarrow (\neg \perp_1 \cup \perp_1)) = (T \Rightarrow (\perp_1 \cup \perp_1)) = (T \Rightarrow \perp_1) = \perp_1$ .

This proves that  $v$  is a counter-model for our formula and

$$\not\models_{\mathbf{L}_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b)).$$

Observe that a  $v$  such that  $v(a) = v(b) = \perp_2$  is also a counter model, as  $v^*((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = ((\perp_2 \Rightarrow \perp_2) \Rightarrow (\neg \perp_2 \cup \perp_2)) = (T \Rightarrow (\perp_2 \cup \perp_2)) = (T \Rightarrow \perp_2) = \perp_2$ .

**QUESTION 4** Write the formula  $A$  from Question 2 as a formula of the language  $\mathcal{L}_{\{\neg, \cup\}}$ , i.e. as a formula  $B$  of  $\mathcal{L}_{\{\neg, \cup\}}$ , such that  $A \equiv B$ . Write down all logical equivalences you need while solving this problem.

**Solution:**

$$\begin{aligned} (((a \wedge \neg c) \Rightarrow \neg b) \cup a) \Rightarrow (c \cup b) &\equiv^{impl} (\neg(((a \wedge \neg c) \Rightarrow \neg b) \cup a) \cup (c \cup b)) \equiv^{impl} (\neg((\neg(a \wedge \neg c) \cup \neg b) \cup a) \cup (c \cup b)) \equiv^{deMorg} \\ &(\neg(((\neg a \cup \neg \neg c) \cup \neg b) \cup a) \cup (c \cup b)) \equiv^{dneg} (\neg((\neg a \cup c) \cup \neg b) \cup a) \cup (c \cup b) \end{aligned}$$

**QUESTION 5**

$S$  is the following proof system:

$$S = (\mathcal{L}_{\{\Rightarrow, \cup, \neg\}}, \mathcal{F}, A1, (r1), (r2))$$

**Axiom**

**A1**  $(A \Rightarrow (A \cup B))$ ,

**Rules** of inference:

$$(r1) \frac{A ; B}{(A \cup \neg B)},$$

$$(r2) \frac{A ; (A \cup B)}{B},$$

1. Verify whether  $S$  is sound/not sound under classical semantics.

**Solution** The system is not sound. Take any  $v$  such that it evaluates  $A = T$  and  $B = F$ . The premiss  $(A \cup B)$  of the rule  $(r2)$  is  $T$  and the conclusion is  $F$ .

2. Find a formal proof of  $\neg(A \Rightarrow (A \cup B))$  in  $S$ , ie. show that

$$\vdash_S \neg(A \Rightarrow (A \cup B)).$$

**Solution** The proof is as follows

$B_1$ :  $(A \Rightarrow (A \cup B))$ , (axiom)

$B_2$ :  $(A \Rightarrow (A \cup B))$ , (axiom)

$B_3$ :  $((A \Rightarrow (A \cup B)) \cup \neg(A \Rightarrow (A \cup B)))$ , (rule r1 application to  $B_1$  and  $B_2$ )

$B_4$ :  $\neg(A \Rightarrow (A \cup B))$ , (rule r2 application to  $B_1$  and  $B_3$ ).

3. Does above point 2. prove that  $\models \neg(A \Rightarrow (A \cup B))$ ?

**Solution** No, the proof used rule  $(r2)$  that is not sound.