CSE371 PRACTICE Midterm Fall 2011

(10 pts + 10 extra)

REAL MIDTERM is TUESDAY, October 25

NAME **ID:** **CS/MAT**

There are 3 questions. Each question would be is 25- 35pts on real Midterm. SOLVE ALL PROBLEMS. ONLY ONE (of my choice) will be corrected. Question 4 is Extra Credit

Ł semantics for $Ł_{\{\neg, \Rightarrow, \cap, \cup\}}$ is defined as follows

Ł Negation

$\neg$	F	$\bot$	T
	T	$\bot$	F

Ł Disjunction

$\cup$	F	$\bot$	T
F	F	$\bot$	T
$\bot$	$\bot$	$\bot$	T
T	T	T	T

Ł Conjunction

$\cap$	F	$\bot$	T
F	F	F	F
$\bot$	F	$\bot$	$\bot$
T	F	$\bot$	T

Ł-Implication

$\Rightarrow$	F	$\bot$	T
F	T	T	T
$\bot$	$\bot$	T	T
T	F	$\bot$	T

QUESTION 1

(1) **Use the fact** that $v : VAR \longrightarrow \{F, \bot, T\}$ be such that $v^*((a \cap b) \Rightarrow \neg b) = \bot$ under **Ł** semantics **to evaluate** $v^*(((b \Rightarrow \neg a) \Rightarrow (a \Rightarrow \neg b)) \cup (a \Rightarrow b))$. Use shorthand notation.

(2) Prove that in classical semantics $\mathcal{L}_{\{\neg,\Rightarrow\}} \equiv \mathcal{L}_{\{\neg,\Rightarrow,\cup\}}$.

(3) Prove that the equivalence $(A \cup B \equiv (\neg A \Rightarrow B)$ defining $\cup$ in classical logic does not hold under **Ł** semantics, but nevertheless $\mathcal{L}_{\{\neg,\Rightarrow\}} \equiv_{\text{Ł}} \mathcal{L}_{\{\neg,\Rightarrow,\cup\}}$.

QUESTION 2 Let H be the following proof system:

$$H = (\ \mathcal{L}_{\{\Rightarrow,\neg\}},\ \mathcal{F},\ AX = \{A1, A2, A3, A4\},\quad MP\ )$$

A1 $(A \Rightarrow (B \Rightarrow A))$,

A2 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$

A4 $(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)$

MP (Rule of inference)

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

(1) Justify that H is SOUND under classical semantics.

(2) Does Deduction Theorem holds for H? Justify shortly your answer.

(3) Justify the fact that H is COMPLETE with respect to all classical semantics tautologies.

(4) Prove that the system H in NOT COMPLETE under the Lukasiewicz semantics **Ł**.

(5) All classical tautologies include for example de Morgan Laws

$$(\neg(A \cup B) \Rightarrow (\neg A \cap \neg B)), \quad (\neg(A \cap B) \Rightarrow (\neg A \cup \neg B))$$

Explain what does it mean that they are provable in H.

(6) Let H' be a proof system obtained from H by adding an additional axiom

A5 $((A \Rightarrow B) \Rightarrow \neg A)$

Is the system H' complete under classical semantics? Justify your answer.

We consider a **sound proof system** (under classical semantics)

$$S = (\ \mathcal{L}_{\{\Rightarrow,\neg\}},\ \ \mathcal{AX},\ \ MP\ ),$$

such that the formulas listed below are provable in S.

1. $(A \Rightarrow (B \Rightarrow A))$,

2. $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

3. $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$,

4. $(A \Rightarrow A)$,

5. $(B \Rightarrow \neg\neg B)$,

6. $(\neg A \Rightarrow (A \Rightarrow B))$,

7. $(A \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B)))$,

8. $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$,

9. $((\neg A \Rightarrow A) \Rightarrow A)$.

The following Lemma holds in S

LEMMA For any $A, B, C \in \mathcal{F}$,

(a) $(A \Rightarrow B), (B \Rightarrow C) \vdash_H (A \Rightarrow C)$,

(b) $(A \Rightarrow (B \Rightarrow C)) \vdash_H (B \Rightarrow (A \Rightarrow C))$.

QUESTION 3

Complete the proof sequence (in S)

$$B_1, ..., B_9$$

by providing comments how each step of the proof was obtained.

$B_1 =\ (A \Rightarrow B)$

$B_2 = (\neg\neg A \Rightarrow A)$
Already PROVED

$B_3 = (\neg\neg A \Rightarrow B)$

$B_4 = (B \Rightarrow \neg\neg B)$

$B_5 = (\neg\neg A \Rightarrow \neg\neg B)$

$B_6 = ((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A))$
ALREADY PROVED

$B_7 = (\neg B \Rightarrow \neg A)$

$B_8 = (A \Rightarrow B) \vdash (\neg B \Rightarrow \neg A)$

$B_9 = ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

HERE IS the Main Definition and Main Lemma needed for the PROOF 1 of the Completeness Theorem for the system S.

Main Definition

Let A be a formula and $b_1, b_2, ..., b_n$ be all propositional variables that occur in A. Let v be variable assignment $v : VAR \longrightarrow \{T, F\}$. We define, for any $A, b_1, b_2, ..., b_n$ and v a corresponding formulas A', $B_1, B_2, ..., B_n$ as follows:

$$A' = \begin{cases} A & \text{if } v^*(A) = T \\ \neg A & \text{if } v^*(A) = F \end{cases}$$

$$B_i = \begin{cases} b_i & \text{if } v(b_i) = T \\ \neg b_i & \text{if } v(b_i) = F \end{cases}$$

for $i = 1, 2, ..., n$.

Main Lemma For any formula A and a variable assignment v, if $A^{'}$, B_1 , B_2, ..., B_n are corresponding formulas defined by the definition stated above, then

$$B_1, B_2, ..., B_n \vdash A'.$$

We write $\vdash\ A$ for $\vdash_S A$ as the system S is fixed.

QUESTION 4 (extra credit 10pts)

We know that the formula

$$A = ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a))$$

is a tautology; i.e. we know that

$$\models\ ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)).$$

Use this information and the method developed in the Proof 1 of Completeness Theorem to show the

$$\vdash\ ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a))$$

Space for Question 4 solution