

CSE371 Practice Midterm SOLUTIONS Fall 2011

L semantics for $\mathcal{L}_{\{\neg, \Rightarrow, \cap, \cup\}}$ is defined as follows

L Negation

\neg	F	\perp	T
	T	\perp	F

L Disjunction

\cup	F	\perp	T
F	F	\perp	T
\perp	\perp	\perp	T
T	T	T	T

L Conjunction

\cap	F	\perp	T
F	F	F	F
\perp	F	\perp	\perp
T	F	\perp	T

L-Implication

\Rightarrow	F	\perp	T
F	T	T	T
\perp	\perp	T	T
T	F	\perp	T

QUESTION 1

- (1) **Use the fact** that $v : VAR \longrightarrow \{F, \perp, T\}$ be such that $v^*((a \cap b) \Rightarrow \neg b) = \perp$ under **L semantics to evaluate** $v^*((b \Rightarrow \neg a) \Rightarrow (a \Rightarrow \neg b)) \cup (a \Rightarrow b)$. Use shorthand notation.

(1) **Solution** : $((a \cap b) \Rightarrow \neg b) = \perp$ in two cases.

C1 $(a \cap b) = \perp$ and $\neg b = F$.

C2 $(a \cap b) = T$ and $\neg b = \perp$.

Case C1: $\neg b = F$, i.e. $b = T$, and hence $(a \cap T) = \perp$ iff $a = \perp$. We get that v is such that $v(a) = \perp$ and $v(b) = T$.

We evaluate: $v^*((b \Rightarrow \neg a) \Rightarrow (a \Rightarrow \neg b)) \cup (a \Rightarrow b) = (((T \Rightarrow \neg \perp) \Rightarrow (\perp \Rightarrow \neg T)) \cup (\perp \Rightarrow T)) = ((\perp \Rightarrow \perp) \cup T) = T$.

Case C2: $\neg b = \perp$, i.e. $b = \perp$, and hence $(a \cap \perp) = T$ what is impossible, hence v from case C1 is the only one.

- (2) Prove that in classical semantics $\mathcal{L}_{\{\neg, \Rightarrow\}} \equiv \mathcal{L}_{\{\neg, \Rightarrow, \cup\}}$.

We define the EQUIVALENCE of LANGUAGES as follows:

Given two languages:

$\mathcal{L}_1 = \mathcal{L}_{CON_1}$ and $\mathcal{L}_2 = \mathcal{L}_{CON_2}$, for $CON_1 \neq CON_2$.

We say that they are **logically equivalent**, i.e.

$$\mathcal{L}_1 \equiv \mathcal{L}_2$$

if and only if the following conditions **C1**, **C2** hold.

C1: For every formula A of \mathcal{L}_1 , there is a formula B of \mathcal{L}_2 , such that

$$A \equiv B,$$

C2: For every formula C of \mathcal{L}_2 , there is a formula D of \mathcal{L}_1 , such that

$$C \equiv D.$$

(2) **Solution:** (Classical case)

C1 holds because any formula of $\mathcal{L}_{\{\neg, \Rightarrow\}}$ is a formula of $\mathcal{L}_{\{\neg, \Rightarrow, \cup\}}$.

C2 holds due to the following definability of connectives equivalence

$$(A \cup B) \equiv (\neg A \Rightarrow B).$$

(3) Prove that the equivalence defining \cup in classical logic does not hold under **L** semantics, but nevertheless $\mathcal{L}_{\{\neg, \Rightarrow\}} \equiv_{\mathbf{L}} \mathcal{L}_{\{\neg, \Rightarrow, \cup\}}$.

Solution $(A \cup B) \not\equiv_{\mathbf{L}} (\neg A \Rightarrow B)$ Take $A = B = \perp$. We get $\perp \cup \perp = \perp$ and $\neg \perp \Rightarrow \perp = \perp \Rightarrow \perp = T$.

Proof that $\mathcal{L}_{\{\neg, \Rightarrow\}} \equiv_{\mathbf{L}} \mathcal{L}_{\{\neg, \Rightarrow, \cup\}}$ holds for **L** semantics.

C1: holds because any formula of $\mathcal{L}_{\{\neg, \Rightarrow\}}$ is a formula of $\mathcal{L}_{\{\neg, \Rightarrow, \cup\}}$.

C2: holds because the definability of connectives equivalence

$$(A \cup B) \equiv ((A \Rightarrow B) \Rightarrow B)$$

holds for **L**. Easy to check by verification.

Observe that the equivalence $(A \cup B) \equiv (\neg A \Rightarrow B)$ defining \cup in terms of \neg and \Rightarrow is a valuable candidate for (**L** semantics definability as the definition of all connectives restricted to T, F is the same as in the classical case. Unfortunately it is not a good one for **L** semantics. It does not prove that other definability equivalence does not exist! Observe that the equivalence $(A \cup B) \equiv (A \Rightarrow B) \Rightarrow B$ provides an alternative proof of **C2** in classical case.

QUESTION 2 Let H be the following proof system:

$$H = (\mathcal{L}_{\{\Rightarrow, \neg\}}, \mathcal{F}, AX = \{A1, A2, A3, A4\}, MP)$$

A1 $(A \Rightarrow (B \Rightarrow A))$,

A2 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$

A4 $((A \Rightarrow B) \Rightarrow A) \Rightarrow A$

MP (Rule of inference)

$$(MP) \frac{A ; (A \Rightarrow B)}{B}$$

(1) Justify that H is SOUND under classical semantics.

Solution Axioms **A1-A3** are axioms of a sound system H_2 , with the same rule MP. So we need only to check if **A4** is sound, i.e. $\models ((A \Rightarrow B) \Rightarrow A) \Rightarrow A$. Assume not, i.e. $((A \Rightarrow B) \Rightarrow A) = T$ and $A = F$. We get $((F \Rightarrow B) \Rightarrow F) = T$. This is impossible, as $(F \Rightarrow B) = T$ for all values of B and $T \Rightarrow F = F$.

(2) Does Deduction Theorem holds for H ? Justify shortly your answer.

Solution Axioms **A1-A2** are axioms of system H_1 for which we proved the Deduction Theorem. System H is a (sound) extension of H_1 and hence the Deduction Theorem holds for it as well.

(3) Justify the fact that H is COMPLETE with respect to all classical semantics tautologies.

Solution Axioms **A1-A3** are axioms of system H_2 for which we proved the Completeness Theorem. System H is a (sound) extension of H_2 and hence the Completeness Theorem holds for it as well.

(4) Prove that the system H is NOT COMPLETE under the Lukasiewicz semantics \mathbf{L} .

Solution System H is not sound under \mathbf{L} semantics. For example axiom **A2** is not \mathbf{L} tautology. $A = \perp, B = \perp, C = F$ evaluates it to \perp . System that is not sound can't be complete.

(5) All classical tautologies include for example de Morgan Laws

$$(\neg(A \cup B) \Rightarrow (\neg A \cap \neg B)), (\neg(A \cap B) \Rightarrow (\neg A \cup \neg B))$$

Explain what does it mean that they are provable in H .

Solution Obviously $\mathcal{L}_{\{\Rightarrow, \neg\}}$ does not contain connectives \cup, \cap and hence de Morgan Laws as written above are not formulas in our language. But we proved that $\mathcal{L}_{\{\Rightarrow, \neg\}} \equiv \mathcal{L}_{\{\Rightarrow, \neg, \cup, \cap\}}$ so the Morgan Laws (as any other formula) expressed in the language $\mathcal{L}_{\{\Rightarrow, \neg\}}$ as an equivalent formula, and then proved.

(6) Let H' be a proof system obtained from H by adding an additional axiom

A5 $((A \Rightarrow B) \Rightarrow \neg A)$

Is the system H' complete under classical semantics? Justify your answer.

Solution H' is not SOUND (axiom **A5** is not a tautology!) hence can't be complete.

We consider a sound proof system (under classical semantics)

$$S = (\mathcal{L}_{\{\Rightarrow, \neg\}}, \mathcal{AX}, MP),$$

such that the formulas listed below are provable in S .

1. $(A \Rightarrow (B \Rightarrow A))$,
2. $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,
3. $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$,
4. $(A \Rightarrow A)$,
5. $(B \Rightarrow \neg \neg B)$,
6. $(\neg A \Rightarrow (A \Rightarrow B))$,
7. $(A \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B)))$,
8. $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$,
9. $((\neg A \Rightarrow A) \Rightarrow A)$.

The following Lemma holds in S

LEMMA For any $A, B, C \in \mathcal{F}$,

- (a) $(A \Rightarrow B), (B \Rightarrow C) \vdash_H (A \Rightarrow C)$,
- (b) $(A \Rightarrow (B \Rightarrow C)) \vdash_H (B \Rightarrow (A \Rightarrow C))$.

QUESTION 3

Complete the proof sequence (in S)

$$B_1, \dots, B_9$$

of

$$((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$$

by providing comments how each step of the proof was obtained.

Solution

$$B_1 = (A \Rightarrow B)$$

Hypothesis

$$B_2 = (\neg\neg A \Rightarrow A)$$

Already Proven

$$B_3 = (\neg\neg A \Rightarrow B)$$

Lemma **a** for $A = \neg\neg A, B = A, C = B$, in B_1, B_2 i.e.

$$(\neg\neg A \Rightarrow A), (A \Rightarrow B) \vdash (\neg\neg A \Rightarrow B)$$

$$B_4 = (B \Rightarrow \neg\neg B)$$

Formula 5

$$B_5 = (\neg\neg A \Rightarrow \neg\neg B)$$

Lemma **a** on B_3, B_4 for $A = \neg\neg A, B = B, C = \neg\neg B$

$$(\neg\neg A \Rightarrow B), (B \Rightarrow \neg\neg B) \vdash (\neg\neg A \Rightarrow \neg\neg B)$$

$$B_6 = ((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A))$$

ALREADY PROVED

$$B_7 = (\neg B \Rightarrow \neg A)$$

B_5, B_6 and MP on B_5, B_6

$$\frac{(\neg\neg A \Rightarrow \neg\neg B); ((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A))}{(\neg B \Rightarrow \neg A)}$$

$$B_8 = (A \Rightarrow B) \vdash (\neg B \Rightarrow \neg A)$$

$B_1 - B_7$

$$B_9 = ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$$

Deduction Theorem on B_8

HERE IS the Main Definition and Main Lemma needed for the PROOF 1 of the Completeness Theorem for the system S .

Main Definition

Let A be a formula and b_1, b_2, \dots, b_n be all propositional variables that occur in A . Let v be variable assignment $v : VAR \rightarrow \{T, F\}$. We define, for any A, b_1, b_2, \dots, b_n and v a corresponding formulas A', B_1, B_2, \dots, B_n as follows:

$$A' = \begin{cases} A & \text{if } v^*(A) = T \\ \neg A & \text{if } v^*(A) = F \end{cases}$$

$$B_i = \begin{cases} b_i & \text{if } v(b_i) = T \\ \neg b_i & \text{if } v(b_i) = F \end{cases}$$

for $i = 1, 2, \dots, n$.

Main Lemma For any formula A and a variable assignment v , if A', B_1, B_2, \dots, B_n are corresponding formulas defined by the definition stated above, then

$$B_1, B_2, \dots, B_n \vdash A'.$$

We write $\vdash A$ for $\vdash_S A$ as the system S is fixed.

QUESTION 4 We know that the formula

$$A = ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a))$$

is a tautology; i.e. we know that

$$\models ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)).$$

Use this information and the method developed in the Proof 1 of Completeness Theorem to show the

$$\vdash ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a))$$

Solution This is a shorter solution than one in the Example in the book; it follows directly the proof 1.

We know that A is a tautology ($\models A$), so $v^*(A) = T$ for all v and $A' = A$ for all v . $A = A(a, b)$, so by the Main Lemma $B_1, B_2 \vdash A$, for B_1, B_2 defined accordingly to v , and $v(a), v(b)$.

Step 1: B_2 elimination. $B_2 = b$ if $v(b) = T$ and $B_2 = \neg b$ if $v(b) = F$.
 For any v such that $v(b) = T$ we get

$$B_1, b \vdash A$$

and for any v such that $v(b) = F$ we get

$$B_1, \neg b \vdash A.$$

By Deduction Theorem we get

(1) $B_1 \vdash (b \Rightarrow A)$ and $B_1 \vdash (\neg b \Rightarrow A)$. We have assumed about the proof system S that for ant formulas A, B ,

$$\vdash ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$$

so in particular

$$\vdash ((b \Rightarrow A) \Rightarrow ((\neg b \Rightarrow A) \Rightarrow A))$$

and by monotonicity

$$B_1 \vdash ((b \Rightarrow A) \Rightarrow ((\neg b \Rightarrow A) \Rightarrow A)).$$

We apply MP twice to (1) and $B_1 \vdash ((b \Rightarrow A) \Rightarrow ((\neg b \Rightarrow A) \Rightarrow A))$ we get that

$$B_1 \vdash A.$$

Step 2: B_1 elimination. $B_1 = a$ if $v(a) = T$ and $B_1 = \neg a$ if $v(a) = F$.
 For any v such that $v(a) = T$ we get

$$a \vdash A$$

and for any v such that $v(a) = F$ we get

$$\neg a \vdash A.$$

By Deduction Theorem we get

(2) $\vdash (a \Rightarrow A)$ and $\vdash (\neg a \Rightarrow A)$. We have assumed about the proof system S that for ant formulas A, B ,

$$\vdash ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$$

so in particular

$$\vdash ((a \Rightarrow A) \Rightarrow ((\neg a \Rightarrow A) \Rightarrow A))$$

We apply MP twice to (2) and $\vdash ((a \Rightarrow A) \Rightarrow ((\neg a \Rightarrow A) \Rightarrow A))$ we get that

$$\vdash A.$$