

Formal Proofs

Example 1

Here are consecutive steps B_1, \dots, B_5 in a proof of

$$(\neg\neg B \Rightarrow B).$$

$$B_1 = ((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$$

$$B_2 = (\neg B \Rightarrow \neg B)$$

$$B_3 = ((\neg B \Rightarrow \neg\neg B) \Rightarrow B)$$

$$B_4 = (\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B))$$

$$B_5 = (\neg\neg B \Rightarrow B)$$

Exercise 1: Complete the proof B_1, \dots, B_5 above by providing comments how each step of the proof was obtained.

Solution

The comments that complete the proof are as follows.

$$B_1 = ((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$$

Axiom A3 for $A = \neg B, B = B$

$$B_2 = (\neg B \Rightarrow \neg B)$$

Lemma 1 for $A = \neg B$

$$B_3 = ((\neg B \Rightarrow \neg\neg B) \Rightarrow B)$$

B_1, B_2 and lemma 2 **b** for $A = (\neg B \Rightarrow \neg\neg B), B = (\neg B \Rightarrow \neg B), C = B$

$$B_4 = (\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B))$$

Axiom A1 for $A = \neg\neg B, B = \neg B$

$$B_5 = (\neg\neg B \Rightarrow B)$$

B_3, B_4 and Lemma 2 **a** for $A = \neg\neg B, B = (\neg B \Rightarrow \neg\neg B), C = B$

General Remark In step B_2, B_3, B_5 we call previously proved Lemmas and use their results. It indicates that if needed we can insert the formal proof of a formula indicated by the lemma.

For example, a completion of steps B_1, B_2, B_3 , as indicated by the lemma 1 is as follows.

We adopt the proof of $(A \Rightarrow A)$ in H_1 to the proof of $(\neg B \Rightarrow \neg B)$ in H_2 by replacing A by $\neg B$. As we insert the proof from the lemma, we rename the "old" step B_3 a B_7 .

$$B_1 = ((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$$

Axiom A3 for $A = \neg B, B = B$

$$B_2 = ((\neg B \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow \neg B)) \Rightarrow ((\neg B \Rightarrow (\neg B \Rightarrow \neg B)) \Rightarrow (\neg B \Rightarrow \neg B))),$$

axiom A2 for $A = \neg B, B = (\neg B \Rightarrow \neg B),$

and $C = \neg B$

$$B_3 = (\neg B \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow \neg B)),$$

axiom A1 for $A = \neg B, B = (\neg B \Rightarrow \neg B)$

$$B_4 = ((\neg B \Rightarrow (\neg B \Rightarrow \neg B)) \Rightarrow (\neg B \Rightarrow \neg B)),$$

MP application to B_4 and B_3

$$B_5 = (\neg B \Rightarrow (\neg B \Rightarrow \neg B)),$$

axiom A1 for $A = \neg B, B = \neg B$

$$B_6 = (\neg B \Rightarrow \neg B)$$

MP application to B_5 and B_4

$B_7 = (" \text{old } B_3) ((\neg B \Rightarrow \neg\neg B) \Rightarrow B)$
 B_1, B_2 and lemma 2**b** for $A = (\neg B \Rightarrow \neg\neg B), B =$
 $(\neg B \Rightarrow \neg B), C = B$

$B_8 = (\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B))$
Axiom A1 for $A = \neg\neg B, B = \neg B$

$B_9 = (\neg\neg B \Rightarrow B)$

B_3, B_4 and Lemma 2 **a** for $A = \neg\neg B, B =$
 $(\neg B \Rightarrow \neg\neg B), C = B$

We repeat the same procedure replacing the "new" B_7 by its formal proof included in the lemma 2 **b**, etc.. until we get a "full" formal proof, if we wished to take time and space to do so. This, of course, is nothing more than the ordinary application of previously proved theorems.

Example 2 Here are consecutive steps B_1, \dots, B_5 in a proof of

$$(B \Rightarrow \neg\neg B).$$

$$B_1 = ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))$$

$$B_2 = (\neg\neg\neg B \Rightarrow \neg B)$$

$$B_3 = ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)$$

$$B_4 = (B \Rightarrow (\neg\neg\neg B \Rightarrow B))$$

$$B_5 = (B \Rightarrow \neg\neg B)$$

Exercise 2: Complete the proof B_1, \dots, B_5 above by providing comments how each step of the proof was obtained.

Solution

The comments that complete the proof B_1, \dots, B_5 are as follows.

$$B_1 = ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))$$

Axiom A3 for $A = B, B = \neg\neg B$

$$B_2 = (\neg\neg\neg B \Rightarrow \neg B)$$

Example 1 for $B = \neg B$

$$B_3 = ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)$$

B_1, B_2 and MP

$$B_4 = (B \Rightarrow (\neg\neg\neg B \Rightarrow B))$$

Axiom A1 for $A = B, B = \neg\neg\neg B$

$$B_5 = (B \Rightarrow \neg\neg B)$$

B_3, B_4 and lemma 2 for

$A = B, B = (\neg\neg\neg B \Rightarrow B), C = \neg\neg B$

Example 3 Here are consecutive steps B_1, \dots, B_{12} in a proof of

$$(\neg A \Rightarrow (A \Rightarrow B)).$$

$$B_1 = \neg A$$

$$B_2 = A$$

$$B_3 = (A \Rightarrow (\neg B \Rightarrow A))$$

$$B_4 = (\neg A \Rightarrow (\neg B \Rightarrow \neg A))$$

$$B_5 = (\neg B \Rightarrow A)$$

$$B_6 = (\neg B \Rightarrow \neg A)$$

$$B_7 = ((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$$

$$B_8 = ((\neg B \Rightarrow A) \Rightarrow B)$$

$$B_9 = B$$

$$B_{10} = \neg A, A \vdash B$$

$$B_{11} = \neg A \vdash (A \Rightarrow B)$$

$$B_{12} = (\neg A \Rightarrow (A \Rightarrow B))$$

Example 6 Prove that

$$\vdash (A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B))))).$$

Solution

1. $A, (A \Rightarrow B) \vdash B$
by MP
2. $A \vdash ((A \Rightarrow B) \Rightarrow B)$
Deduction Theorem
3. $\vdash (A \Rightarrow ((A \Rightarrow B) \Rightarrow B))$
Deduction Theorem

4. $\vdash (((A \Rightarrow B) \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B)))$

Example 5 for $A = (A \Rightarrow B), B = B$

5. $\vdash (A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B))))$

3. and 4. and lemma 2a for $A = A, B = ((A \Rightarrow B) \Rightarrow B), C = (\neg B \Rightarrow \neg A)$

Example 7 Here are consecutive steps B_1, \dots, B_{12} in a proof of

$$((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B)).$$

$$B_1 = (A \Rightarrow B)$$

$$B_2 = (\neg A \Rightarrow B)$$

$$B_3 = ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$$

$$B_4 = (\neg B \Rightarrow \neg A)$$

$$B_5 = ((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg\neg A))$$

$$B_6 = (\neg B \Rightarrow \neg\neg A)$$

$$B_7 = ((\neg B \Rightarrow \neg\neg A) \Rightarrow ((\neg B \Rightarrow \neg A) \Rightarrow B))$$

$$B_8 = ((\neg B \Rightarrow \neg A) \Rightarrow B)$$

$$B_9 = B$$

$$B_{10} = (A \Rightarrow B), (\neg A \Rightarrow B) \vdash B$$

$$B_{11} = (A \Rightarrow B) \vdash ((\neg A \Rightarrow B) \Rightarrow B)$$

$$B_{12} = ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$$