INTRODUCTION EXERCISES on SETS
SOLUTIONS

QUESTION 1 Use the above definition to prove the following

FACT 1 A set A is INFINITE iff it contains a countably infinite subset, i.e. one can define a $1-1$ sequence $\{a_n\}_{n\in N}$ of some elements of A.

SOLUTION 1. Implication $\rightarrow$

If A is infinite, then we can define a 1-1 sequence of elements of A.
Let A be infinite,
We define a sequence

$$a_1, \dots, a_n, \dots$$

as follows.
1. Observe that $A \neq \emptyset$,because if $A = \emptyset$, A would be finite. contradiction.
So there is an element of $a \in A$.
We define

$$a_1 = a$$

2.Consider a set $A - \{a_a\} = A_1$. $A_1 \neq \emptyset$ because if $A = \emptyset$, then $A - \{a_1\} = \emptyset$ and A is Finite.Contradiction.
So there is an element $a_2 \in A - \{a_1\}$ and $a_1 \neq a_2$.
We defined

$$a_1, a_2$$

3.Assume now that we have defined an n-elements and sequence

$$a_{1,2}, \dots, a_n \text{ for } a_1 \neq a_2 \neq \dots \neq a_n$$

Consider a set $A_n = A - \{a_1, \dots, a_n\}$.
The set $A_n \neq \emptyset$ because if $A - \{a_1, \dots, a_n\} = \emptyset$,then A is finite.Contradiction
So there is an element

$$a_{n+1} \in A - \{a_1, \dots, a_n\}$$

and $a_{n+1} \neq a_n \neq \cdots \neq a_1$
By mathematical induction,
we have defined a 1-1 sequence

$$a_1, a_2, \dots, a_n, \dots$$

elements of A.
2. Implication $\leftarrow$
If A contain a 1-1 sequence, then A is infinite.
Assume A is not infinite; i.e A is finite. Every subset of finite set is finite, so we can't have a 1-1 infinite sequence of elements of A. Contradiction.

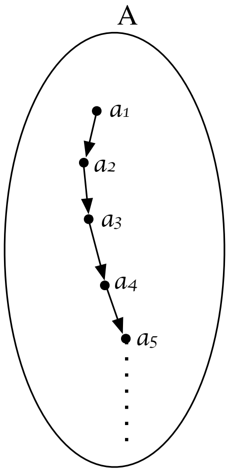

Figure 1: problem 1

QUESTION 2 Use the above definition and FACT 1 from Question 1 to prove the following characterization of infinite sets.

Dedekind Theorem A set A is INFINITE iff there is a set proper subset B of the set A such that $|A| = |B|$.

SOLUTION Part1. If A is infinite, then there is $B \subsetneq A$ and

$$f : A \xrightarrow[onto]{1-1} B$$

A is infinite, by Q1, we have a 1-1 sequence

$$a_1, a_2, \ldots, a_n, \ldots$$

of elements A.
We take $B = A - \{a_1\}$, $B \subsetneq A$ and we define a function

$$f : A \xrightarrow[onto]{1-1} B$$

as follows

$$\begin{gathered} f(a_1) = a_2 \\ f(a_2) = a_3 \\ \vdots \\ f(a_n) = a_{n+1} \\ f(a) = a, \text{for all other } a \in A \end{gathered}$$

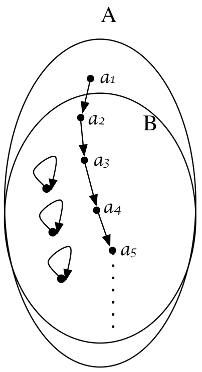

Figure 2: problem 2:part 1

obviously, f is 1-1,onto
Observe: we have other choises of B!.
Part 2. Assume that we have $B \subsetneq A$ are

$$f : A \xrightarrow[onto]{1-1} B$$

We use Q1 to show that A is infinite; i.e we construct an 1-1 sequence $a_1 \ldots a_n$ of elements of A_n as follows.
$B \subsetneq A$,so $A - B \neq \emptyset$ and we have $b \in A - B$. This is our first element of the sequence.
Observe: $f : A \xrightarrow[onto]{1-1} B$, so $f(b) \in B$ and $b \in A - B$, hence $f(b) \neq b$ and $f(b)$ is our second element of the sequence.
We have now,
$b, f(b) \qquad f(b) \neq b, b \in A - B, f(b) \leftarrow B$
Take new,
$ff(b)$.As f is 1-1 and $f(b) \neq b$, we get $ff(b) \neq f(b) \neq b, ff(b) \in B$ and the sequence $b, f(b), ff(b)$ is 1-1.
We create $ff(b) = f^2(b)$
We continue the construction by mathematical induction.
Assume that we have constructed a 1-1 sequence

$$b, f(b), f^{(}b), f^3(b), \ldots, f^n(b)$$

Observe that $ff^n(b) = f^{n+1}(b) \neq f^n(b)$ as f is 1-1.
By mathematical induction, we have that $\{f^n(b)\}_{n \in N}$ is a 1-1 sequence of elements of A and hence A is infinite.

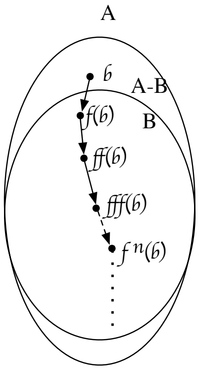

Figure 3: problem 2:part 2

QUESTION 3 Use technique from DEDEKIND THEOREM to prove the following

Theorem For any infinite set A and its finite subset B, $|A| = |A - B|$.

SOLUTION A is infinite, then by Q1 there is a 1-1 sequence:

$$a_1, a_2, \ldots, a_n, \ldots$$

of elements of A.
Let $|B| = k$. We choose k 1-1 sequence $\{C_n^k\}_{n \in N}$ of the sequence $\{a_n\}_{n \in N}$, such that $C_n^j \neq C_n^i$ for all $j \neq i, 1 \leq i, j \leq k$ and all $n \in N$.
Let $B = \{b_1, \ldots, b_k\}$.We construct a function $f : A \xrightarrow[onto]{1-1} A - \{b_1, \ldots, b_k\}$ as follows

$$\begin{array}{ll} f(b_1) = c_1^1, & f(c_1^1) = c_2^1, \ldots, f(c_n^1) = c_{n+1}^1 \\ f(b_2) = c_1^2, & f(c_1^2) = c_2^2, \ldots, f(c_n^2) = c_{n+1}^2 \\ & \vdots \\ f(b_k) = c_1^k, & f(c_1^k) = c_2^k, \ldots, f(c_n^k) = c_{n+1}^k \end{array}$$
$$f(a) = a \text{ all } a \in A - B$$

As all sequences $\{C_n^m\}_{n \in N, m=1,\ldots,k}$ are 1-1, and different, the function f is 1-1 and obviously ONTO $A - B$.

QUESTION 4 Use DEDEKIND THEOREM to prove that the set N of natural numbers is infinite.

SOLUTION We use Dedekind theorem i.e we must define $f : N \xrightarrow[onto]{1-1} B \subsetneq N$. There are many such function for example $f(n) = n + 1.f : N \xrightarrow[onto]{1-1} N - \{0\}$
One can also use Q1 and define any 1-1 sequences in N.

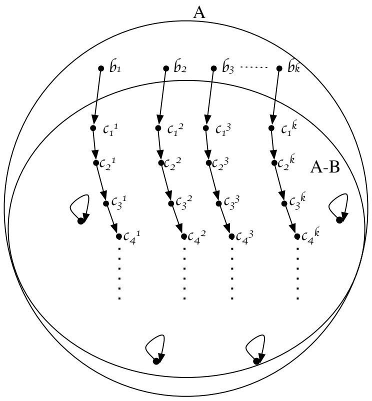

Figure 4: problem 3

QUESTION 5 Use DEDEKIND THEOREM to prove that the set R of real numbers is infinite.

SOLUTION We use Dedekind theorem

$$f(x) = 2^x \qquad x \in R$$

$$f : R \xrightarrow[onto]{1-1} R^+$$

One can also use Q1 and define any 1-1 sequences in R.

QUESTION 6 Use technique from DEDEKIND THEOREM to prove that the interval $[a, b], a < b$ of real numbers is infinite and that $|[a, b]| = |(a, b)|$.

SOLUTION1 Use construction in the proof of Q3.
$f : [a, b] \xrightarrow[onto]{1-1} [a, b] - \{a, b\} = (a, b)$
This is the soution I had in mine!

SOLUTION2 Use Q3 $(a, b) = [a, b] - B, B$:finite

QUESTION 7 Prove, using the above definitions 3 and 4 that for any cardinal numbers $\mathcal{M}, \mathcal{N}, \mathcal{K}$ the following formulas hold:

$$1.\mathcal{N} \le \mathcal{N}$$

$$2. If \;\; \mathcal{N} \le \mathcal{M} \;\; and \;\; \mathcal{M} \le \mathcal{K}, \;\; then \;\; \mathcal{N} \le \mathcal{K}.$$

SOLUTION 1. $\mathcal{N} \le \mathcal{N}$ means that for any set A, $|A| \le |A|$
$f(a) = a$ establishes $f : A \xrightarrow{1-1} A$
2. $\mathcal{N} \le \mathcal{M}$ *and* $\mathcal{M} \le \mathcal{K}$, *then* $\mathcal{N} \le \mathcal{K}$.
We have $|A| = \mathcal{N}$,$|b| = \mathcal{M}$,$|C| = \mathcal{K}$ and $f : A \xrightarrow{1-1} B$ and $g : B \xrightarrow{1-1} C$,then we have to construct $h : A \xrightarrow{1-1} C$.

h is a composition of f and g. i.e $h(a) = g(f(a))$, all $a \in A$

QUESTION 8 Prove, for any sets A, B, C the following holds.

Fact 2

$$If \quad C \subseteq B \subseteq A \; and \; |A| = |C|, \quad then \quad |A| = |B| = |C|.$$

To prove $|A| = |B|$ you must use definition 3, i.e to construct a proper function. Use the construction from proofs of Fact 1 and Question 3

SOLUTION 1. A, B, C are finite and $|A| = |C|$, and $C \subseteq B \subseteq A$,so $A = B = C$, and have $|A| = |B| = |C|$

2.A, B, C are infinite sets, we have $|A| = |C|$ i.e we have $f : A \xrightarrow[onto]{1-1} C$

We want to construct a function

$$g : A \xrightarrow[onto]{1-1} B, \text{ where } A \subseteq B \subseteq C$$

Take $A - B$. We assume that $A - B \neq \emptyset$, if not, $A = B$, and $|A| = |C|$ given $|A| = |B| = |C|$.

We consider case $C \subset B \subset A$. Take any $a \in (A - B)$, as $f : A \xrightarrow[onto]{1-1} C, f(a) \in C$,$f$ is 1-1 so $ff(a) \neq f(a)$ $\ldots$ in general $f^n(a) \neq f^{n+1}(a)$ and we have a sequence for any $a \in A - B$

$f(a), f^2(a), \ldots, f^n(a) \ldots$ of elements of C.

We construct a function $g : A \xrightarrow[onto]{1-1} B$

$$g(a) = f(a)$$
$$g(f(a)) = f^2(a)$$
$$g(f^2(a)) = f^3(a))$$
$$\vdots$$
$$g(f^n(a)) = f^{n+1}(a)$$
$$g(x) = x \qquad \text{for all other } x \in A$$

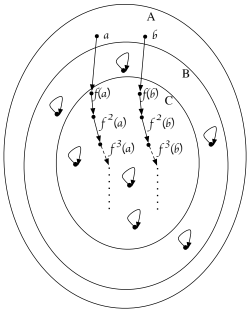

Figure 5: problem 8: Figure of function $g : A \xrightarrow[onto]{1-1} B$. a,b represent any two element of A

QUESTION 9 Prove the following

Berstein Theorem (1898) For any cardinal numbers $\mathcal{M},\mathcal{N}$

$$\mathcal{N} \leq \mathcal{M} \ \ and \ \ \mathcal{M} \leq \mathcal{N} \ \ then \mathcal{N} = \mathcal{M}.$$

SOLUTION Let A,B be two sets such that $|A| = \mathcal{N}$,$|B| = \mathcal{M}$, we rewrite on theorem as

Berstein Theorem For any sets A,B
If $|A| \leq |B|$ and $|B| \leq |A|$,then $|A| = |B|$

case1. The sets A, B are disjoint.

As $|A| \leq |B|$, we have a function $f : A \xrightarrow{1-1} B$,i.e $f : A \xrightarrow[onto]{1-1} fA \subseteq B$ and $|A| = |fA|$ where fA denotes the image of A under f.

As $|B| \leq |A|$, we have a function $g : B \xrightarrow[onto]{1-1} gB \subseteq A$ and $|B| = |gB|$

We picture it as follow.

$|B| = |gB|, |A| = |fA|$

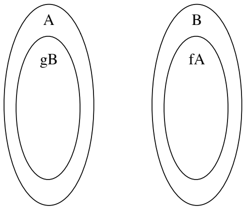

Figure 6: problem 9

As $f : A \xrightarrow{1-1} B$ and $gB \subseteq A$, we get $fgB \subseteq fA$ and hence

$$fgB \subseteq fA \subseteq B \tag{1}$$

Also, $gB \subseteq A$ and $g : B \xrightarrow{1-1} B$. Hence,$fg : B \xrightarrow[onto]{1-1} fgB$ and

$$|B| = |fgB| \tag{2}$$

We have a following picture.

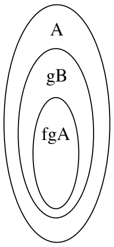

Figure 7: problem 9

By eq.2, $|B| = |fgB|$ and by eq.1, $fgB \subseteq fA \subseteq B$ and $|B| = |fA|$
By Q8, we get

$$|fA| = |B|$$

Hence,$|B| = |A|$

case2. the set A, B are NOT disjoint.

Repeat the same(or Google the proof) for the following picture.

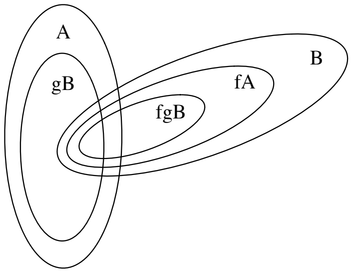

Figure 8: problem 9