Chapter 10: Introduction to Intuitionistic Logic

PART 2: Hilbert Proof System for propositional intuitionistic logic.

Language is a propositional language

$$\mathcal{L} = \mathcal{L}_{\{\cup, \cap, \Rightarrow, \neg\}}$$

with the set of formulas denoted by $\mathcal{F}$.

Axioms

A1 $((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$,

A2 $(A \Rightarrow (A \cup B))$,

A3 $(B \Rightarrow (A \cup B))$,

A4 $((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \cup B) \Rightarrow C)))$,

A5 $((A \cap B) \Rightarrow A)$,

A6 $((A \cap B) \Rightarrow B)$,

A7 $((C \Rightarrow A) \Rightarrow ((C \Rightarrow B) \Rightarrow (C \Rightarrow (A \cap B)))$,

A8 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C))$,

A9 $(((A \cap B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))$,

A10 $(A \cap \neg A) \Rightarrow B)$,

A11 $((A \Rightarrow (A \cap \neg A)) \Rightarrow \neg A)$,

where A, B, C are any formulas in $\mathcal{L}$.

Rules of inference: we adopt a Modus Ponens rule

$$(\mathbf{MP})\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

as the only inference rule.

A proof system I

$$I = (\ \mathcal{L},\ \mathcal{F}, \mathbf{A1} - \mathbf{A11},\ (\mathbf{MP})\ ),$$

for $\mathcal{L}$, **A1 - A11** defined above, is called Hilbert Style Formalization for Intuitionistic Propositional Logic.

This set of axioms is due to Rasiowa (1959). It differs from Heyting original set of axioms but they are equivalent.

We introduce, as usual, the notion of a formal proof in I and denote by

$$\vdash_I A$$

the fact that A has a formal proof in I, or that that A is *intuitionistically provable*.

We write

$$\models_I A$$

to denote that the formula A is intuitionistic tautology.

Completeness Theorem for I For any formula $A \in \mathcal{F}$,

$$\vdash_I A \text{ ı } and\ only\ if\ \models_I A.$$

The Completeness Theorem gives us the right to replace the notion of a theorem (provable formula) of a given intuitionistic proof system by an independent of the proof system and more intuitive (as we all have some notion of truthfulness) notion of the intuitionistic tautology.

The intuitionistic logic has been created as a rival to the classical one. So a question about the relationship between these two is a natural one.

The following classical tautologies are provable in I and hence, by the Completeness Theorem, are also intuitionistic tautologies.

1. $(A \Rightarrow A)$,

2. $(A \Rightarrow (B \Rightarrow A))$,

3. $(A \Rightarrow (B \Rightarrow (A \cap B)))$,

4. $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

5. $(A \Rightarrow \neg\neg A)$,

6. $\neg(A \cap \neg A)$,

7. $((\neg A \cup B) \Rightarrow (A \Rightarrow B))$,

8. $(\neg(A \cup B) \Rightarrow (\neg A \cap \neg B))$,

9. $((\neg A \cap \neg B) \Rightarrow (\neg(A \cup B))$,

10. $((\neg A \cup \neg B) \Rightarrow (\neg(A \cap B))$,

11. $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$,

12. $((A \Rightarrow \neg B) \Rightarrow (B \Rightarrow \neg A))$,

13. $(\neg\neg\neg A \Rightarrow \neg A)$,

14. $(\neg A \Rightarrow \neg\neg\neg A)$,

15. $(\neg\neg(A \Rightarrow B) \Rightarrow (A \Rightarrow \neg\neg B))$,

16. $((C \Rightarrow A) \Rightarrow ((C \Rightarrow (A \Rightarrow B)) \Rightarrow (C \Rightarrow B)))$.

Examples of classical tautologies that are not intuitionistic tautologies

17. $(A \cup \neg A)$,

18. $(\neg\neg A \Rightarrow A)$,

19. $((A \Rightarrow B) \Rightarrow (\neg A \cup B))$,

20. $(\neg(A \cap B) \Rightarrow (\neg A \cup \neg B))$,

21. $((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow A))$,

22. $((\neg A \Rightarrow \neg B) \Rightarrow (B \Rightarrow A))$,

23. $(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)$.

Connections between Classical and Intuitionistic logics.

The first connection is quite obvious. Let us observe that if we add the axiom

A12 $(A \cup \neg A)$

to the set of axioms of the system I we obtain a complete Hilbert proof system C for the classical logic.

This proves the following.

Theorem 1 Every formula that is derivable intuitionistically is classically derivable, i.e.

$$if \quad \vdash_I A, \quad then \quad \vdash A$$

where we use symbol $\vdash$ for classical (complete classical proof system) provability.

By the Completeness Theorem we get the following.

Theorem 2 For any formula $A \in \mathcal{F}$,

$$if \quad \models_I A, \quad then \quad \models \; A.$$

The next relationship shows how to obtain intuitionistic tautologies from the classical tautologies and vice versa.

The following relationships were proved by Glivenko in 1929 and independently, in a semantic form by Tarski in 1938.

Theorem 3 (Glivenko) For any formula $A \in \mathcal{F}$, A is a classically provable if and only if $\neg\neg A$ is an intuitionistically provable, i.e.

$$\vdash_I A \quad iff \quad \vdash \neg\neg A$$

where we use symbol $\vdash$ for classical (complete classical proof system) provability.

Theorem 4 (Tarski) For any formula $A \in \mathcal{F}$, A is a classical tautology if and only if $\neg\neg A$ is an intuitionistic tautology, i.e.

$$\models A \;\; if \; and \; only \; if \;\; \models_I \neg\neg A.$$

The following relationships were proved by Gödel in 1331.

Theorem 5 (Gödel) For any $A, B \in \mathcal{F}$, a formula $(A \Rightarrow \neg B)$ is a classically provable if and only if it is an intuitionistically provable, i.e.

$$\vdash \ (A \Rightarrow \neg B) \ \ if \ and \ only \ if \ \ \vdash_I (A \Rightarrow \neg B).$$

Theorem 6 (Gödel) If a formula A contains no connectives except $\cap$ and $\neg$, then A is a classically provable if and only if it is an intuitionistically provable tautology.

By the Completeness Theorems for classical and intuitionisctic logics we get the following equivalent semantic form of theorems 5 and 6.

Theorem 7 For any $A, B \in \mathcal{F}$, a formula $(A \Rightarrow \neg B)$ is a classical tautology if and only if it is an intuitionistic tautology, i.e.

$$\models (A \Rightarrow \neg B) \ \ if\ and\ only\ if\ \ \models_I (A \Rightarrow \neg B).$$

Theorem 8 If a formula A contains no connectives except $\cap$ and $\neg$, then A is a classical tautology if and only if it is an intuitionistic tautology.

On intuitionistically derivable disjunctions.

In a classical logic it is possible for the disjunction $(A \cup B)$ to be a tautology when neither A nor B is a tautology. The tautology $(A \cup \neg A)$ is the simplest example. This does not hold for the intuitionistic logic.

Theorem 9 (stated without the proof by Gödel in 1910)

For any $A, B \in \mathcal{F}$, a formula $(A \cup B)$ is intuitionistically provable if and only if A is intuitionistically provable or B is intuitionistically provable i.e.

$$\vdash \ (A \cup B) \ \ if\ and\ only\ if \ \ \vdash_I A, \ \ or \ \ \vdash_I B.$$

Theorem 9 was proved by Gentzen in 1935 via his proof system **LI** which is presented and discussed in the next chapter.

We obtain, via the Completeness Theorem the following equivalent semantic version of the above.

Theorem 10 (Tarski) For any $A, B \in \mathcal{F}$, a disjunction $(A \cup B)$ is intuitionistic tautology if and only if either A or B is intuitionistic tautology, i.e.

$$\models_I (A \cup B) \quad \textit{if and only if} \quad \models_I A \;\; \textit{or} \;\; \models_I B.$$