Chapter 11 (Part 2)
Gentzen Sequent Calculus GL

The proof system GL for the classical propositional logic is a version of the original Gentzen (1934) systems **LK**.

A constructive proof of the completeness theorem for the system **GL** is very similar to the proof of the completeness theorem for the system **RS**.

Expressions of the system like in the original Gentzen system **LK** are Gentzen *sequents*.

Hence we use also a name Gentzen sequent calculus.

Language of GL: $\mathcal{L} = \mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg,\}}$.

We add a new symbol to the alphabet: $\longrightarrow$. It is called a Gentzen arrow.

The sequents are built out of finite sequences (empty included) of formulas, i.e. elements of $\mathcal{F}^*$, and the additional sign $\longrightarrow$.

We denote, as in the **RS** system, the finite sequences of formulas by Greek capital letters Γ, Δ, Σ, with indices if necessary.

Sequent definition: a sequent is the expression

$$\Gamma \longrightarrow \Delta,$$

where $\Gamma, \Delta \in \mathcal{F}^*$.

Meaning of sequents Intuitively, we interpret a sequent

$$A_1, ..., A_n \longrightarrow B_1, ..., B_m,$$

where $n, m \geq 1$ as a formula

$$(A_1 \cap ... \cap A_n) \Rightarrow (B_1 \cup ... \cup B_m).$$

The sequent: $A_1, ..., A_n \longrightarrow$ (where $n \geq 1$) means *that* $A_1 \cap ... \cap A_n$ *yields a contradiction.*

The sequent $\longrightarrow B_1, ..., B_m$ (where $m \geq 1$) means $\models (B_1 \cup ... \cup B_m)$.

The empty sequent: $\longrightarrow$ means *a contradiction.*

Given non empty sequences: Γ, Δ, we denote by

$$\sigma_{\Gamma}$$

any conjunction of all formulas of Γ, and by

$$\delta_{\Delta}$$

any disjunction of all formulas of Δ.

The intuitive semantics (meaning, interpretation) of a sequent $\Gamma \longrightarrow \Delta$ (where Γ, Δ are nonempty) is

$$\Gamma \longrightarrow \Delta \equiv (\sigma_{\Gamma} \Rightarrow \delta_{\Delta}).$$

Formal semantics for sequents (expressions of **GL**)

Let $v : VAR \longrightarrow \{T, F\}$ be a truth assignment, v^* its (classical semantics) extension to the set of formulas $\mathcal{F}$.

We extend v^* to the set

$$SEQ = \{ \ \Gamma \longrightarrow \Delta : \ \Gamma, \Delta \in \mathcal{F}^* \ \}$$

of all sequents as follows: for any sequent $\Gamma \longrightarrow \Delta \in SEQ$,

$$v^*(\Gamma \longrightarrow \Delta) \ = \ v^*(\sigma_\Gamma) \Rightarrow v^*(\delta_\Delta).$$

In the case when $\Gamma = \emptyset$ or $\Delta = \emptyset$ we define:

$$v^*(\ \longrightarrow \Delta) \ = \ (T \ \Rightarrow v^*(\delta_\Delta)),$$

$$v^*(\Gamma \longrightarrow \ ) \ = \ (v^*(\sigma_\Gamma) \Rightarrow \ F).$$

The sequent $\Gamma \longrightarrow \Delta$ is *satisfiable* if there is a truth assignment $v : VAR \longrightarrow \{T, F\}$ such that $v^*(\Gamma \longrightarrow \Delta) = T$.

Model for $\Gamma \longrightarrow \Delta$ is any v, such that

$$v^*(\Gamma \longrightarrow \Delta) = T.$$

We write it

$$v \models \Gamma \longrightarrow \Delta$$

Counter- model is any v such that

$$v^*(\Gamma \longrightarrow \Delta) = F.$$

We write it

$$v \not\models \Gamma \longrightarrow \Delta.$$

Tautology is any sequent $\Gamma \longrightarrow \Delta$, such that $v^*(\Gamma \longrightarrow \Delta) = T$ for all truth assignments $v : VAR \longrightarrow \{T, F\}$, i.e.

$$\models \Gamma \longrightarrow \Delta.$$

Example Let $\Gamma \longrightarrow \Delta$ be a sequent

$$a, (b \cap a) \longrightarrow \neg b, (b \Rightarrow a).$$

The truth assignment v for which $v(a) = T$ and $v(b) = T$ is a model for $\Gamma \longrightarrow \Delta$, as shows the following computation.

$v^*(a, (b{\cap}a) \longrightarrow \neg b, (b \Rightarrow a)) = v^*(\sigma_{\{a,(b\cap a)\}}) \Rightarrow v^*(\delta_{\{\neg b,(b\Rightarrow a)\}}) = v(a) \cap (v(b) \cap v(a)) \Rightarrow \neg v(b) \cup (v(b) \Rightarrow v(a)) = T \cap TcapT \Rightarrow \neg T \cup (T \Rightarrow T) = T \Rightarrow (F \cup T) = T \Rightarrow T = T.$

Observe that the only v for which $v^*(\Gamma) = v^*(a, (b \cap a) = T$ is the above $v(a) = T$ and $v(b) = T$ that is a model for $\Gamma \longrightarrow \Delta$.

It is impossible to find v which would falsify it, what proves that

$$\models \; a, (b \cap a) \;\longrightarrow\; \neg b, (b \Rightarrow a).$$

Definition of GL

Axioms of GL: Any sequent of variables (positive literals) which contains a propositional variable that appears on both sides of the sequent arrow $\longrightarrow$, i.e any sequent of the form

$$\Gamma'_1, a, \Gamma'_2 \ \longrightarrow \ \Delta'_1, a, \Delta'_2,$$

for any $a \in VAR$ and any sequences $\Gamma'_1, \Gamma'_2, \Delta'_1, \Delta'_2 \in VAR^*$.

Inference rules of GL

We denote by $\Gamma^{'}$, $\Delta^{'}$ finite sequences formed out of literals i.e. out of propositional variables or negations of propositional variables. Γ, Δ denote any finite sequences of formulas.

Conjunction rules

$$(\cap \rightarrow)\ \frac{\Gamma', A, B, \Gamma \ \longrightarrow \ \Delta'}{\Gamma', (A \cap B), \Gamma \ \longrightarrow \ \Delta'},$$

$$(\rightarrow \cap)\ \frac{\Gamma \ \longrightarrow \ \Delta, A, \Delta' \ ; \ \Gamma \ \longrightarrow \ \Delta, B, \Delta'}{\Gamma \ \longrightarrow \ \Delta, (A \cap B), \Delta'},$$

Disjunction rules

$$(\rightarrow \cup)\ \frac{\Gamma \ \longrightarrow \ \Delta, A, B, \Delta'}{\Gamma \ \longrightarrow \ \Delta, (A \cup B), \Delta'},$$

$$(\cup \rightarrow)\ \frac{\Gamma', A, \Gamma \ \longrightarrow \ \Delta' \ ; \ \Gamma', B, \Gamma \ \longrightarrow \ \Delta'}{\Gamma', (A \cup B), \Gamma \ \longrightarrow \ \Delta'},$$

Implication rules

$$(\longrightarrow\Rightarrow)\ \frac{\Gamma', A, \Gamma \ \longrightarrow\ \Delta, B, \Delta'}{\Gamma', \Gamma \ \longrightarrow\ \Delta, (A \Rightarrow B), \Delta'},$$

$$(\Rightarrow\longrightarrow)\ \frac{\Gamma', \Gamma \ \longrightarrow\ \Delta, A, \Delta' \ ; \ \Gamma', B, \Gamma \ \longrightarrow\ \Delta, \Delta'}{\Gamma', (A \Rightarrow B), \Gamma \ \longrightarrow\ \Delta, \Delta'},$$

Negation rules

$$(\neg \longrightarrow)\ \frac{\Gamma', \Gamma \ \longrightarrow\ \Delta, A, \Delta'}{\Gamma', \neg A, \Gamma \ \longrightarrow\ \Delta, \Delta'},$$

$$(\longrightarrow \neg)\ \frac{\Gamma', A, \Gamma \ \longrightarrow\ \Delta, \Delta'}{\Gamma', \Gamma \ \longrightarrow\ \Delta, \neg A, \Delta'}.$$

We define:

$$\mathbf{GL} = (SEQ, AL, (\cup), (\neg\cup), (\cap), (\neg\cap), (\Rightarrow), (\neg\Rightarrow), (\neg\neg))$$

Formal proof of a sequent $\Gamma \longrightarrow \Delta$ in the proof system **GL** we understand any sequence

$$\Gamma_1 \longrightarrow \Delta_1,\ \Gamma_2 \longrightarrow \Delta_2,\ ....,\ \Gamma_n \longrightarrow \Delta_n$$

of sequents of formulas (elements of SEQ), such that $\Gamma_1 \longrightarrow \Delta_1 \in AL$, $\Gamma_n \longrightarrow \Delta_n = \Gamma \longrightarrow \Delta$, and for all i $(1 < i \leq n)$ $\Gamma_i \longrightarrow \Delta_i \in AL$, or $\Gamma_i \longrightarrow \Delta_i$ is a conclusion of one of the inference rules of **GL** with all its premisses placed in the sequence $\Gamma_1 \longrightarrow \Delta_1,\ ....\Gamma_{i-1} \longrightarrow \Delta_{i-1}$.

We write, as usual,

$$\vdash_{\mathbf{GL}} \Gamma \longrightarrow \Delta$$

to denote that $\Gamma \longrightarrow \Delta$ has a formal proof in **GL**.

A formula $A \in \mathcal{F}$, has a proof in if the sequent $\longrightarrow A$ has a proof in **GL**, i.e. we define:

$$\vdash_{\mathbf{GL}} A \quad iff \quad \longrightarrow \ A.$$

A proof tree, or **GL**-proof of $\Gamma \longrightarrow \Delta$ is a tree

$$\mathbf{T}_{\Gamma \longrightarrow \Delta}$$

of sequents satisfying the following conditions:

1. The topmost sequent, i.e *the root* of $\mathbf{T}_{\Gamma \longrightarrow \Delta}$ is $\Gamma \longrightarrow \Delta$,

2. *All leafs* are axioms,

3. *The nodes* are sequents such that each sequent on the tree follows from the ones immediately preceding it by one of the rules.

We write the proof- trees indicating additionally the name of the inference rule used at each step of the proof.

Remark Proof search, i.e. **decomposition tree** for a given formula A and hence a **proof** of A in **GL** is not always unique!!

For example, a tree-proof (in **GL**) of the de Morgan law

$$(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

is the following.

$$\longrightarrow \; (\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

$$\mid (\longrightarrow\Rightarrow)$$

$$\neg(a \cap b) \; \longrightarrow \; (\neg a \cup \neg b)$$

$$\mid (\longrightarrow \cup)$$

$$\neg(a \cap b) \; \longrightarrow \; \neg a, \neg b$$

$$\mid (\longrightarrow \neg)$$

$$b, \neg(a \cap b) \; \longrightarrow \; \neg a$$

$$\mid (\longrightarrow \neg)$$

$$b, a, \neg(a \cap b) \; \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$b, a \longrightarrow \; (a \cap b)$$

$$\bigwedge(\longrightarrow \cap)$$

$$b, a \longrightarrow \; a \qquad\qquad b, a \longrightarrow \; b$$

Exercise 1 : Write all other proofs of $\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$ in **GL**.

Exercise 2: Verify that the axiom and the rules of inference of **GL** are sound, i.e. that the following theorem holds.

Soundness Theorem for GL: For any sequent $\Gamma \longrightarrow \Delta \in SEQ$,

$$if \quad \vdash_{\mathbf{GL}} \Gamma \longrightarrow \Delta \quad then \quad \models \Gamma \longrightarrow \Delta.$$

Completeness Theorem For any sequent $\Gamma \longrightarrow \Delta \in SEQ$,

$$\vdash_{\mathbf{GL}} \Gamma \longrightarrow \Delta \quad iff \quad \models \Gamma \longrightarrow \Delta.$$

The proof of the Completeness Theorem is similar to the proof for the **RS** system and is assigned as an exercise.