CHAPTER 11: Automated Proof Systems (3)

RS: Counter Models Generated by Decomposition Trees

RS: Proof of COMPLETENESS THEOREM

Counter-model generated by the decomposition tree.

Example: Given a formula A:

$$((a \Rightarrow b) \cap \neg c) \cup (a \Rightarrow c))$$

and its decomposition tree $\mathbf{T}_A$.

$$(((a \Rightarrow b) \cap \neg c) \cup (a \Rightarrow c))$$

$$\mid (\cup)$$

$$((a \Rightarrow b) \cap \neg c), (a \Rightarrow c)$$

$$\bigwedge (\cap)$$

$$(a \Rightarrow b), (a \Rightarrow c) \qquad \neg c, (a \Rightarrow c)$$

$$\mid (\Rightarrow) \qquad\qquad \mid (\Rightarrow)$$

$$\neg a, b, (a \Rightarrow c) \qquad \neg c, \neg a, c$$

$$\mid (\Rightarrow)$$

$$\neg a, b, \neg a, c$$

Consider a non-axiom leaf:

$$\neg a, b, \neg a, c$$

Let v be any variable assignment

$$v: \ VAR \ \longrightarrow \ \{T, F\}$$

such that it makes this non-axiom leaf FALSE, i.e. we put

$$v(a) = T, v(b) = F, v(c) = F.$$

Obviously, we have that

$$v^*(\neg a, b, \neg a, c) = \neg T \cup F \cup \neg T \cup F = F.$$

Moreover, all our rules of inference are **sound** (to be proven formally in the next section).

Rules soundness means that if one of premisses of a rule is FALSE, so is the conclusion.

Hence, the soundness of the rules proves (by induction on the degree of sequences $\Gamma \in \mathbf{T}_A$) that v, as defined above **falsifies all sequences** on the branch of $\mathbf{T}_A$ that ends with the non-axiom leaf $\neg a, b, \neg a, c$.

In particular, the formula A is on this branch, hence

$$v^*(((a \Rightarrow b) \cap \neg c) \cup (a \Rightarrow c)) = F$$

and v is **a counter-model** for A.

The truth assignments defined by a non- axiom leaves are called **counter-models generated** by the decomposition tree.

The construction of the counter-models generated by the decomposition trees are crucial to the proof of the Completeness Theorem for **RS**.

F "climbs" the Tree $\mathbf{T}_A$.

$$\mathbf{T}_A$$

$$(((a \Rightarrow b) \cap \neg c) \cup (a \Rightarrow c)) = \mathbf{F}$$

$$\mid (\cup)$$

$$((a \Rightarrow b) \cap \neg c), (a \Rightarrow c) = \mathbf{F}$$

$$\bigwedge (\cap)$$

$$(a \Rightarrow b), (a \Rightarrow c) = \mathbf{F} \qquad \neg c, (a \Rightarrow c)$$

$$\mid (\Rightarrow) \qquad\qquad \mid (\Rightarrow)$$

$$\neg a, b, (a \Rightarrow c) = \mathbf{F} \qquad \neg c, \neg a, c$$

$$\mid (\Rightarrow)$$

$$\neg a, b, \neg a, c = \mathbf{F}$$

Observe that the same construction applies to any other non-axiom leaf, if exists, and gives the other "F climbs the tree" picture, and hence other counter- model for A.

By the Uniqueness of the Decomposition Tree Theorem all possible counter-models (restricted) for A are those generated by the non- axioms leaves of the $\mathbf{T}_A$. In our case the formula A has only one non-axiom leaf, and hence only one (restricted) counter model.

RS: COMPLETENESS THEOREM

We prove first the Soundness Theorem for RS; and then the completeness part of the Completeness Theorem.

Soundness Theorem 1

For any $\Gamma \in \mathcal{F}^*$,

if $\vdash_{\mathbf{RS}} \Gamma$, then $\models \Gamma$.

Proof: we prove here as an example the soundness of two of inference rules. We leave the proof for the other rules as an exercise.

We show that rules $(\Rightarrow)$ and $(\neg\cup)$ of **G are sound**.

We show even more, i.e. that the premisses and conclusion of both rules are **logically equivalent**.

I.e. that for all v, $v^*(Premiss(es)) = T$, implies that $v^*(Conclusion) = T$.

We hence show the following.

Equivalency: If $P_1, (P_2)$ are premiss(es) of any rule of RS, C is its conclusion, then $v^*(P_1) = v^*(C)$ in case of one premiss rule and $v^*(P_1) \cap v^*(P_2) = v^*(C)$, in case of the two premisses rule.

Consider the rule $(\cup)$.

$$(\cup) \quad \frac{\Gamma', A, B, \Delta}{\Gamma', (A \cup B), \Delta}.$$

By the definition:

$v^*(\Gamma', A, B, \Delta) = v^*(\delta_{\{\Gamma', A, B, \Delta\}}) = v^*(\Gamma') \cup v^*(A) \cup v^*(B) \cup v^*(\Delta) = v^*(\Gamma') \cup v^*(A \cup B) \cup v^*(\Delta) = v^*(\delta_{\{\Gamma', (A \cup B), \Delta\}}) = v^*(\Gamma', (A \cup B), \Delta)$.

Consider the rule $(\neg\cup)$.

$$(\neg\cup) \quad \frac{\Gamma^{'}, \neg A, \Delta \ \ : \ \ \Gamma^{'}, \neg B, \Delta}{\Gamma^{'}, \neg(A \cup B), \Delta}.$$

By the definition:

$v^*(\Gamma^{'}, \neg A, \Delta) \cap v^*(\Gamma^{'}, \neg B, \Delta) = (v^*(\Gamma^{'}) \cup v^*(\neg A) \cup v^*(\Delta)) \cap (v^*(\Gamma^{'}) \cup v^*(\neg B) \cup v^*(\Delta)) = (v^*(\Gamma^{'}, \Delta) \cup v^*(\neg A)) \cap (v^*(\Gamma^{'}, \Delta) \cup v^*(\neg B)) =$ by distributivity $= (v^*(\Gamma^{'}, \Delta) \cup (v^*(\neg A) \cap v^*(\neg B)) = v^*(\Gamma^{'}) \cup v^*(\Delta) \cup (v^*(\neg A \cap \neg B)) =$ by the logical equivalence of $(\neg A \cap \neg B)$ and $\neg(A \cup B) = v^*(\delta_{\{\Gamma', \neg(A \cup B), \Delta\}} = v^*(\Gamma^{'}, \neg(A \cup B), \Delta))$.

Proofs for all other rules follow the above pattern (and proper logical equivalencies).

From the above **Soundness Theorem 1** we get as a corollary, in a case when Γ is a one formula sequence, the following soundness lemma for formulas.

Soundness Theorem 2

For any $A \in \mathcal{F}$,

if $\vdash_{\mathbf{RS}} A$, then $\models A$.

Now we are ready to prove the Completeness Theorem, in two forms: sequence, and formula.

Completeness Theorem 1

For any formula $A \in \mathcal{F}$,

$\vdash_{\mathbf{RS}} A$ if and only if $\models A$.

Completeness Theorem 2

For any $\Gamma \in \mathcal{F}^*$,

$\vdash_{\mathbf{RS}} \Gamma$ if and only if $\models \Gamma$.

Both proofs are carried by proving the contraposition implication to the Completeness Part, as the soundness part has been already proven.

Proof: as an example, we list the main steps in the proof of a contraposition of the **Completeness Theorem 1**.

$$\text{If} \quad \nvdash_{\mathbf{RS}} A, \quad \text{then} \quad \not\models A.$$

To prove the Completenes Theorem we proceed as follows.

Define, for each $A \in \mathcal{F}$ its decomposition tree (**Decomposition Tree Definition**).

Prove that the decomposition tree is finite unique (**Decomposition Tree Theorem**) and has the following property:

$\vdash_{\mathbf{G}} A$ iff all leafs of the decomposition tree of A are axioms.

What means that if $\not\vdash_{\mathbf{RS}} A$, then there is a leaf L of the decomposition tree of A, which is not an axiom.

Observe, that by soundness, if one premiss of a rule of **RS** is FALSE, so is the conclusion.

Hence by soundness and the definition of the decomposition tree any truth assignment v that falsifies an non axiom leaf, i.e. any v such that $v^*(L) = F$ falsifies A, namely $v^*(A) = F$ and hence constitutes a counter model for A. This ends that proof that $\not\models A$.

Essential part:

Given a formula A such that $\not\vdash_{\mathbf{RS}} A$ and its decomposition tree of A with a non-axiom leaf L.

We define **a counter-model** v determined by the non- axiom leat L as follows:

$$v(a) = \begin{cases} F & \text{if a appears in } L \\ T & \text{if } \neg a \text{ appears in } L \\ any\ value & \text{if a does not appear in } L \end{cases}$$

This proves that $\not\models A$ and ends the proof of the **Completeness Theorem** for RS.