Chapter 12: Gentzen Sequent Calculus for Intuitionistic Logic

Part 1: LI System

The proof system LI was published by Gentzen in 1935 as a particular case of his proof system **LK** for the classical logic.

We discussed a version of the original Gentzen's system **LK** in the previous chapter.

We present now the proof system **LI** and then we show how it can be extended to the original Gentzen system **LK**.

Language of LI

We consider the set of all Gentzen sequents built out of the formulas of our language $\mathcal{L}$ and the additional symbol $\longrightarrow$, as defined in the previous section:

$$SEQ = \{\ \Gamma \longrightarrow \Delta :\ \Gamma, \Delta \in \mathcal{F}^*\ \}.$$

In the intuitionistic logic we deal only with sequents of the form

$$\Gamma\ \longrightarrow\ \Delta,$$

where Δ consists of at most one formula.

The intuitionistic sequents are elements of a following subset IS of the set SEQ of all sequents.

$ISEQ = \{\Gamma\ \longrightarrow\ \Delta :$ Δ consists of at most one formula $\}$.

Axioms of LI consists of any sequent from the set $ISEQ$ which contains a formula that appears on both sides of the sequent arrow $\longrightarrow$, i.e any sequent of the form

$$\Gamma_1, A, \Gamma_2 \;\longrightarrow\; A.$$

Inference rules of LI

The set inference rules is divided into two groups: the structural rules and the logical rules.

Structural Rules of **LI**

Weakening

$$(\to weakening)\ \ \frac{\Gamma\ \longrightarrow}{\Gamma\ \longrightarrow\ A}$$

A **is called** the weakening formula.

Contraction

$$(contr \rightarrow) \quad \frac{A, A, \Gamma \;\longrightarrow\; \Delta}{A, \Gamma \;\longrightarrow\; \Delta}$$

A **is called** the contraction formula.

Δ **contains** at most one formula.

Exchange

$$(exchange \rightarrow) \quad \frac{\Gamma_1, A, B, \Gamma_2 \;\longrightarrow\; \Delta}{\Gamma_1, B, A, \Gamma_2 \;\longrightarrow\; \Delta},$$

Δ **contains** at most one formula.

Logical Rules of LI

Conjunction rules

$$(\cap \rightarrow) \quad \frac{A, B, \Gamma \longrightarrow \Delta}{(A \cap B), \Gamma \longrightarrow \Delta},$$

$$(\rightarrow \cap) \quad \frac{\Gamma \longrightarrow A \; ; \; \Gamma \longrightarrow B}{\Gamma \longrightarrow (A \cap B)},$$

Disjunction rules

$$(\rightarrow \cup)_1 \quad \frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow (A \cup B)},$$

$$(\rightarrow \cup)_1 \quad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow (A \cup B)},$$

$$(\cup \rightarrow) \quad \frac{A, \Gamma \longrightarrow \Delta \; ; \; B, \Gamma \longrightarrow \Delta}{(A \cup B), \Gamma \longrightarrow \Delta},$$

Δ **contains** at most one formula.

Implication rules

$$(\longrightarrow\Rightarrow) \quad \frac{A, \Gamma \;\longrightarrow\; B}{\Gamma \;\longrightarrow\; (A \Rightarrow B)},$$

$$(\Rightarrow\longrightarrow) \quad \frac{\Gamma \;\longrightarrow\; A \;;\; B, \Gamma \;\longrightarrow\; \Delta}{(A \Rightarrow B), \Gamma \;\longrightarrow\; \Delta},$$

Δ **contains** at most one formula.

Negation rules

$$(\neg\longrightarrow) \quad \frac{\Gamma \;\longrightarrow\; A}{\neg A, \Gamma \;\longrightarrow\;},$$

$$(\longrightarrow\neg) \quad \frac{A, \Gamma \;\longrightarrow\;}{\Gamma \;\longrightarrow\; \neg A}.$$

We define

$\mathbf{LI} = (\mathcal{L}, ISEQ,\ AL$, Structural rules, Logical rules $\})$.

LK - Original Gentzen system for the classical propositional logic.

Language of LK: $\mathcal{L} = \mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$, and $\mathcal{E} = SEQ$, for

$$SEQ = \{\Gamma \longrightarrow \Delta : \quad \Gamma, \Delta \in \mathcal{F}^*\}.$$

Axioms of LK: any sequent of the form

$$\Gamma_1, A, \Gamma_2 \longrightarrow \Gamma_3, A, \Gamma_4.$$

Rules of inference of LK

1. We adopt all rules of **LI** with no intuitionistic restriction that the sequence Δ in the succedent of the sequence is at most one formula.

2. We add two structural rules to the system **LI**.

We add one more contraction rule:

$$(\rightarrow contr) \quad \frac{\Gamma \longrightarrow \Delta, A, A,}{\Gamma \longrightarrow \Delta, A},$$

We add one more exchange rule:

$$(\rightarrow exchange) \quad \frac{\Delta \longrightarrow \Gamma_1, A, B, \Gamma_2}{\Delta \longrightarrow \Gamma_1, B, A, \Gamma_2}.$$

Observe that they both become obsolete in **LI** .

The rules of inference of **LK** are hence as follows.

Structural Rules of LK

Weakening

$$(weakening \rightarrow) \quad \frac{\Gamma \quad \longrightarrow \quad \Delta}{A, \Gamma \quad \longrightarrow \quad \Delta},$$

$$(\rightarrow weakening) \quad \frac{\Gamma \quad \longrightarrow}{\Gamma \quad \longrightarrow \quad \Delta, A} \; .$$

A **is called** the weakening formula.

Contraction

$$(contr \rightarrow) \quad \frac{A, A, \Gamma \; \longrightarrow \; \Delta}{A, \Gamma \; \longrightarrow \; \Delta},$$

$$(\rightarrow contr) \quad \frac{\Gamma \; \longrightarrow \; \Delta, A, A,}{\Gamma \; \longrightarrow \; \Delta, A},$$

A **is called** the contraction formula.

Exchange

$$(exchange \rightarrow) \quad \frac{\Gamma_1, A, B, \Gamma_2 \; \longrightarrow \; \Delta}{\Gamma_1, B, A, \Gamma_2 \; \longrightarrow \; \Delta},$$

$$(\rightarrow exchange) \quad \frac{\Delta \longrightarrow \Gamma_1, A, B, \Gamma_2}{\Delta \longrightarrow \Gamma_1, B, A, \Gamma_2}.$$

Logical Rules of LK

Conjunction rules

$$(\cap \rightarrow) \quad \frac{A, B, \Gamma \longrightarrow \Delta}{(A \cap B), \Gamma \longrightarrow \Delta},$$

$$(\rightarrow \cap) \quad \frac{\Gamma \longrightarrow \Delta,\ A \ ;\ \Gamma \longrightarrow \Delta,\ B\Delta}{\Gamma \longrightarrow \Delta, (A \cap B)}.$$

Disjunction rules

$$(\rightarrow \cup) \quad \frac{\Gamma \longrightarrow \Delta, A, B}{\Gamma \longrightarrow \Delta, (A \cup B)},$$

$$(\cup \rightarrow) \quad \frac{A, \Gamma \longrightarrow \Delta \ ;\ B, \Gamma \longrightarrow \Delta}{(A \cup B), \Gamma \longrightarrow \Delta}.$$

Implication rules

$$(\longrightarrow\Rightarrow)\quad \frac{A, \Gamma \ \longrightarrow\ \Delta, B}{\Gamma \ \longrightarrow\ \Delta, (A \Rightarrow B)},$$

$$(\Rightarrow\longrightarrow)\quad \frac{\Gamma \ \longrightarrow\ \Delta, A \ ;\ B, \Gamma \ \longrightarrow\ \Delta}{(A \Rightarrow B), \Gamma \ \longrightarrow\ \Delta}.$$

Negation rules

$$(\neg \longrightarrow)\quad \frac{\Gamma \ \longrightarrow\ \Delta,\ A}{\neg A, \Gamma \ \longrightarrow\ \Delta},$$

$$(\longrightarrow \neg)\quad \frac{A, \Gamma \ \longrightarrow\ \Delta}{\Gamma \ \longrightarrow\ \Delta, \neg A}.$$

We define formally $\mathbf{LK} = (\mathcal{L}, , SEQ,\ AL,$ Structural rules, Logical rules).