Chapter 12: Gentzen Sequent Calculus for Intuitionistic Logic

PART 2: Examples of proof search decomposition trees in LI

Search for proofs in **LI** is a much more complicated process then the one in classical logic systems **RS** or **GL**.

Proof search procedure consists of building the decomposition trees.

Remark 1: in **RS** the decomposition tree $\mathbf{T}_A$ of any formula A is always unique.

Remark 2: in **GL** the "blind search" defines, for any formula A a finite number of decomposition trees, but it can be proved that the search can be reduced to examining only one of them, due to the absence of structural rules.

Remark 3: In **LI** the structural rules play a vital role in the proof construction and hence, in the proof search.

The fact that a given decomposition tree ends with an axiom leaf does not always imply that the proof does not exist. It might only imply that our search strategy was not good.

The problem of deciding whether a given formula A does, or does not have a proof in **LI** becomes more complex then in the case of Gentzen system for classical logic.

Example 1

Determine whether

$$\vdash_{\mathbf{LI}} \longrightarrow A$$

for $A = ((\neg A \cap \neg B) \Rightarrow \neg(A \cup B))$.

If we find a decomposition tree such that all its leaves are axiom, we have a proof.

If all possible decomposition trees have a non-axiom leaf, proof of A in **LI** does not exist.

Consider the following decomposition tree

$\mathbf{T1_A}$

$$\longrightarrow ((\neg A \cap \neg B) \Rightarrow (\neg(A \cup B))$$

$$\mid (\longrightarrow\Rightarrow)$$

$$(\neg A \cap \neg B) \longrightarrow \neg(A \cup B)$$

$$\mid (\longrightarrow \neg)$$

$$(A \cup B), (\neg A \cap \neg B) \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$(\neg A \cap \neg B), (A \cup B) \longrightarrow$$

$$\mid (\cap \longrightarrow)$$

$$\neg A, \neg B, (A \cup B) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\neg B, (A \cup B) \longrightarrow A$$

$$\mid (\longrightarrow weak)$$

$$\neg B(A \cup B) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$(A \cup B) \longrightarrow B$$

$$\bigwedge(\cup \longrightarrow)$$

$$A \longrightarrow B \qquad\qquad B \longrightarrow B$$

$$non-axiom \qquad\qquad axiom$$

The tree $\mathbf{T1_A}$ has a non-axiom leaf, so it does not constitute a proof in **LI**.

Observe that the decomposition tree in **LI** is not always unique.

Hence this fact does not yet prove that proof doesn't exist.

Let's consider now the following tree

$$\mathbf{T2_A}$$

$$\longrightarrow ((\neg A \cap \neg B) \Rightarrow (\neg(A \cup B))$$

$$\mid (\longrightarrow \Rightarrow)$$

$$(\neg A \cap \neg B) \longrightarrow \neg(A \cup B)$$

$$\mid (\longrightarrow \neg)$$

$$(A \cup B), (\neg A \cap \neg B) \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$(\neg A \cap \neg B), (A \cup B) \longrightarrow$$

$$\mid (\cap \longrightarrow)$$

$$\neg A, \neg B, (A \cup B) \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$\neg A, (A \cup B), \neg B \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$(A \cup B), \neg A, \neg B \longrightarrow$$

$$\bigwedge(\cup \longrightarrow)$$

$$A, \neg A, \neg B \longrightarrow \qquad B, \neg A, \neg B \longrightarrow$$

$$\mid (exch \longrightarrow) \qquad \mid (exch \longrightarrow)$$

$$\neg A, A, \neg B \longrightarrow \qquad B, \neg B, \neg A \longrightarrow$$

$$\mid (\neg \longrightarrow) \qquad \mid (exch \longrightarrow)$$

$$A, \neg B \longrightarrow A \qquad \neg B, B, \neg A \longrightarrow$$

$$axiom \qquad \mid (\neg \longrightarrow)$$

$$B, \neg A \longrightarrow B$$

$$axiom$$

All leaves of $\mathbf{T2_A}$ are axioms, what proves that $\mathbf{T2_A}$ is a proof of A.

Hence we proved that

$$((\neg A \cap \neg B) \Rightarrow \neg(A \cup B)).$$

Example 2: Proof that

Part 1

$$\vdash_{\mathbf{LI}} \longrightarrow (A \Rightarrow \neg\neg A),$$

Part 2

$$\not\vdash_{\mathbf{LI}} \longrightarrow (\neg\neg A \Rightarrow A).$$

Solution of Part 1: We construct some, or all decomposition trees of

$$\longrightarrow (A \Rightarrow \neg\neg A).$$

The tree that ends with all axioms leaves is a proof of $(A \Rightarrow \neg\neg A)$ in **LI**.

Consider the following decomposition tree of $\longrightarrow A$, for $A = (A \Rightarrow \neg\neg A)$..

$$\mathbf{T}_A$$

$$\longrightarrow (A \Rightarrow \neg\neg A).$$

$$\mid (\longrightarrow\Rightarrow)$$

$$A \longrightarrow \neg\neg A$$

$$\mid (\longrightarrow \neg)$$

$$\neg A, A \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$A \longrightarrow A$$

$$axiom$$

All leaves of $\mathbf{T}_A$ are axioms what proves that $\mathbf{T}_A$ is a proof of $\longrightarrow (A \Rightarrow \neg\neg A)$.

We don't need to construct other decomposition trees.

Solution of Part 2: in order to prove that

$$\not\vdash_{\mathbf{LI}} \longrightarrow (\neg\neg A \Rightarrow A)$$

we have to construct all decomposition trees of

$$(\longrightarrow A \Rightarrow \neg\neg A)$$

and show that each of them has an non-axiom leaf.

Decomposition trees construction is as follows.

$$\mathbf{T1}_A$$

$$\longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow\Rightarrow)$$

$$one\ of\ 2\ choices$$

$$\neg\neg A \longrightarrow A$$

$$\mid (\longrightarrow weak)$$

$$one\ of\ 2\ choices$$

$$\neg\neg A \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$one\ of\ 2\ choices$$

$$\longrightarrow \neg A$$

$$\mid (\longrightarrow \neg)$$

$$one\ of\ 2\ choices$$

$$A \longrightarrow$$

$$non-axiom$$

Another tree is:

$$\mathbf{T2}_A$$

$$\longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow\Rightarrow)$$

$$one\ of\ 2\ choices$$

$$\neg\neg A \longrightarrow A$$

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$$\mid (contr \longrightarrow)$$

$$second\ of\ 2\ choices$$

$$\neg\neg A, \neg\neg A \longrightarrow A$$

$$\mid (\longrightarrow weak)$$

$$first\ of\ 2\ choices$$

$$\neg\neg A, \neg\neg A \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$first\ of\ 2\ choices$$

$$\neg\neg A \longrightarrow \neg A$$

$$\mid (\longrightarrow \neg)$$

$$the\ only\ choice$$

$$A, \neg\neg A \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$the\ only\ choice$$

$$\neg\neg A, A \longrightarrow$$

$$| \ (\longrightarrow \neg)$$
$$the \ only \ choice$$
$$A \longrightarrow \neg A$$
$$| \ (\longrightarrow \neg)$$
$$first \ of \ 2 \ choices$$
$$A, A \longrightarrow$$
$$non - axiom$$

We can see from the above decomposition trees that the "blind" construction of all possible trees only leads to more complicated trees.

This is due to the presence of structural rules.

Observe that the "blind" application of the rule $(contr \rightarrow)$ gives an infinite number of decomposition trees.

In order to decide that none of them will produce a proof we need some extra knowledge about patterns of their construction, or just simply about the number useful of application of structural rules within the proofs.

In this case we can just make an "external" observation that the our first tree $\mathbf{T1}_A$ is in a sense a minimal one; that all other trees would only complicate this one in an inessential way, i.e. we will never produce a tree with all axioms leaves.

One can formulate a deterministic procedure giving a finite number of trees, but the proof of its correctness require some extra knowledge.

Within the scope of this book we accept the "external" explanation as a sufficient solution, provided it is correct.

As we can see from the above examples structural rules and especially the $(contr \longrightarrow)$ rule complicates the proof searching task.

Both Gentzen type proof systems **RS** and **GL** from the previous chapter don't contain the structural rules.

They also are complete with respect to classical semantics.

The original Gentzen system **LK** which does contain the structural rules is also complete.

Hence, all three classical proof system **RS, GL, LK** are equivalent.

This proves that the structural rules can be eliminated from the system **LK**.

A natural question of elimination of structural rules from the intutionistic Gentzen system **LI** arizes.

The following example illustrates the negative answer.

Example 3 We know, by the theorem about the connection between classical and intuitionistic logic and corresponding Completeness Theorems that

For any formula $A \in \mathcal{F}$,

$$\models\ A \ \ if\ and\ only\ if\ \ \vdash_I \neg\neg A,$$

$\models\ A$ means that A is a classical tautology,

$\vdash_I$ means that A is intutionistically provable, i.e. is provable in any intuitionistically complete proof system.

The system LI is intuitionistically complete, so we have that for any formula A,

$$\models\ A \ \ if\ and\ only\ if\ \ \vdash_{\mathbf{LI}} \neg\neg A.$$

Obviously $\models (\neg\neg A \Rightarrow A)$, so we know that

$$\neg\neg(\neg\neg A \Rightarrow A)$$

must have a proof in **LI**.

We are going to prove the structural rule

$$(contr \longrightarrow)$$

is essential to the existence of its proof.

The formula $\neg\neg(\neg\neg A \Rightarrow A)$ is not provable in **LI** without the rule $(contr \longrightarrow)$.

The following decomposition tree is a proof of $A = \neg\neg(\neg\neg A \Rightarrow A)$ in **LI**.

$$\mathbf{T}_A$$

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \neg)$$

one of 2 *choices*

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (contr \longrightarrow)$$

one of 2 *choices*

$$\neg(\neg\neg A \Rightarrow A), \neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

one of 2 *choices*

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow\Rightarrow)$$

one of 3 *choices*

$$\neg(\neg\neg A \Rightarrow A), \neg\neg A \longrightarrow A$$

$$\mid (\longrightarrow weak)$$

one of 2 *choices*

$$\neg(\neg\neg A \Rightarrow A), \neg\neg A \longrightarrow$$

$$\mid (exch \longrightarrow)$$

one of 3 *choices*

$$\neg\neg A, \neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$one\ of\ 3\ choices$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow \neg A$$

$$\mid (\longrightarrow \neg)$$

$$one\ of\ 3\ choices$$

$$A, \neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$one\ of\ 2\ choices$$

$$\neg(\neg\neg A \Rightarrow A), A \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$one\ of\ 3\ choices$$

$$A \longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \Rightarrow)$$

$$one\ of\ 3\ choices$$

$$\neg\neg A, A \longrightarrow A$$

$$axiom$$

Assume now that the rule $(contr \longrightarrow)$ is not available.

All possible decomposition trees are as follows.

$$\mathbf{T1}_A$$

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \neg)$$

$$one\ of\ 2\ choices$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$only\ one\ choice$$

$$\longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \Rightarrow)$$

$$one\ of\ 2\ choices$$

$$\neg\neg A \longrightarrow A$$

$$\mid (\longrightarrow weak)$$

$$only\ one\ choice$$

$$\neg\neg A \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$only\ one\ choice$$

$$\longrightarrow \neg A$$

$$\mid (\longrightarrow \neg)$$

$$one\ of\ 2\ choices$$

$$A \longrightarrow$$

$$non - axiom$$

Next one is

$$\mathbf{T2}_A$$

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow weak)$$

$$second\ of\ 2\ choices$$

$$\longrightarrow$$

$$non-axiom$$

And the next is

$$\mathbf{T3}_A$$

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \neg)$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow weak)$$

$$second\ of\ 2\ choices$$

$$\longrightarrow$$

$$non - axiom$$

And the last one is

$$\mathbf{T4}_A$$

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$
$$\mid (\longrightarrow \neg)$$
$$\neg(\neg\neg A \Rightarrow A) \longrightarrow$$
$$\mid (\neg \longrightarrow)$$
$$\longrightarrow (\neg\neg A \Rightarrow A)$$
$$\mid (\longrightarrow\Rightarrow)$$
$$]$$
$$\neg\neg A \longrightarrow A$$
$$\mid (\longrightarrow weak)$$
$$only\ one\ choice$$
$$\neg\neg A \longrightarrow$$
$$\mid (\neg \longrightarrow)$$
$$only\ one\ choice$$
$$\longrightarrow \neg A$$
$$\mid (\longrightarrow weak)$$
$$second\ of\ 2\ choices$$
$$\longrightarrow$$
$$non - axiom$$

This proves that the formula $\neg\neg(\neg\neg A \Rightarrow A)$ is not provable in $\mathbf{LI}$ without $(contr \longrightarrow)$ rule, i.e. that this rule can't be eliminated.