CHAPTER 13

Gentzen Style Proof System for Classical Predicate Logic - The System QRS

Part One

System QRS Definition

Let $\mathcal{F}$ denote a set of formulas of a Predicate (first Order) Logic Language

$$\mathcal{L}(\mathbf{P}, \mathbf{F}, \mathbf{C}) = \mathcal{L}_{\{\cap, \cup, \Rightarrow, \neg\}}(\mathbf{P}, \mathbf{F}, \mathbf{C})$$

for **P, F, C** countably infinite sets of predicate, functional, and constant symbols respectively.

The rules of inference of our system **QRS** will operate, as in the propositional case, on **finite sequences of formulas**, i.e. elements of $\mathcal{F}^*$, instead of just plain formulas $\mathcal{F}$, as in Hilbert style formalizations.

We will denote the sequences of formulas by Γ, Δ, Σ, with indices if necessary.

Intuitive semantics If Γ is a sequence

$$A_1, A_2, ..., A_n$$

then by δ_Γ we will understand the disjunction of all formulas of Γ.

As we know, the disjunction in classical logic is commutative, i.e., for any formulas A, B, C, $A \cup (B \cup C) \equiv (A \cup B) \cup C$, we w will denote any of those formulas by

$$A \cup B \cup C = \delta_{\{A,B,C\}}.$$

Similarly, we will write

$$\delta_\Gamma = A_1 \cup A_2 \cup ..., \cup A_n.$$

The sequence Γ is said to be **satisfiable** (**falsifiable**) if the formula $\delta_\Gamma = A_1 \cup A_2 \cup ..., \cup A_n$ is satisfiable (falsifiable).

The sequence Γ is said to be **a tautology** if the formula $\delta_\Gamma = A_1 \cup A_2 \cup ..., \cup A_n$ is a tautology.

The system QRS consists of **two axiom** schemas and **eleven rules** of inference.

The rules form two groups.

First group is similar to the propositional case and called

Each rule of this group introduces a new logical connective or its negation, so we will name them, as in the propositional case: $(\cup), (\neg\cup), (\cap), (\neg\cap), (\Rightarrow), (\neg \Rightarrow)$, and $(\neg\neg)$.

The second group deals with the quantifiers. It consists of four rules.

Two quantifiers rules introduce the universal and existential quantifiers, and are named $(\forall)$ and $(\exists)$, respectively.

The two other rules correspond to the De Morgan Laws and deal with the negation of the universal and existential quantifiers, and ere named $(\neg\forall)$ and $(\neg\exists)$, respectively.

As the axioms we adopt any sequence which contains any formula and its negation, i.e any sequence of the form

$$\Gamma_1, A, \Gamma_2, \neg A, \Gamma_3$$

or of the form

$$\Gamma_1, \neg A, \Gamma_2, A, \Gamma_3,$$

for any formula $A \in \mathcal{F}$ and any sequences of formulas $\Gamma_1, \Gamma_2, \Gamma_3 \in \mathcal{F}^*$.

We will denote the axioms by

$$\mathcal{AX}^*.$$

QRS proof system is defined as

$$\mathbf{QRS} = (\mathcal{F}^*, \mathcal{AX}^*, (\cup), (\neg\cup), (\cap), (\neg\cap),$$

$$(\Rightarrow), (\neg \Rightarrow), (\neg\neg), (\neg\forall), (\neg\exists), (\forall), (\exists))$$

QRS system is called a **Gentzen- style formalization** of classical predicate calculus.

In order to define the rules of inference of **QRS** we need to introduce some definitions. They are straightforward modification of the corresponding definitions for the propositional logic.

We form, as in the propositional case, a special subset

$$\mathcal{LIT} \subseteq \mathcal{F}$$

of formulas, called a set of all **literals**, which is defined now as follows.

$$\mathcal{LIT} = \{A \in \mathcal{F} : A \in \mathcal{AF}\} \cup \{\neg A \in \mathcal{F} : A \in \mathcal{AF}\},$$

where $\mathcal{AF} \subseteq \mathcal{F}$ is the set of all **atomic (elementary) formulas** of the first order language, i.e.

$$\mathcal{AF} = \{P(t_1, ..., t_n) : P \in \mathbf{P} \ \}$$

$P \in \mathbf{P}$ is any n-argument predicate symbol, and $t_i \in T$ are terms.

The elements of the set

$$\{A \in \mathcal{F} : A \in \mathcal{AF}\}$$

are called **positive literals** and the elements of the set

$$\{\neg A \in \mathcal{F} : A \in \mathcal{AF}\}$$

are called **negative literals**.

I.e atomic (elementary) formulas are called positive literals and the negation of an atomic (elementary) formula is called a negative literal.

Indecomposable formulas Literals are also called the indecomposable formulas.

Now we form **finite sequences** out of formulas (and, as a special case, out of literals). We need to distinguish the sequences formed out of literals from the sequences formed out of other formulas, so we adopt exactly the same notation as in the propositional case.

We denote by $\Gamma^{'}$, $\Delta^{'}$, $\Sigma^{'}$ finite sequences (empty included) formed out of **literals** i.e. out of the elements of $\mathcal{LIT}$ i.e. we assume that

$$\Gamma^{'}, \Delta^{'}, \Sigma^{'} \in \mathcal{LIT}^{*}.$$

We denote by Γ, Δ, Σ the elements of $\mathcal{F}^{*}$ i.e the finite sequences (empty included) formed out of elements of $\mathcal{F}$.

We define the inference rules of **QRS** as follows.

Group 1: Propositional Inference rules

Disjunction rules

$$(\cup)\ \frac{\Gamma', A, B, \Delta}{\Gamma', (A \cup B), \Delta}, \qquad (\neg\cup)\ \frac{\Gamma', \neg A, \Delta \ :\ \Gamma', \neg B, \Delta}{\Gamma', \neg(A \cup B), \Delta}$$

Conjunction rules

$$(\cap)\ \frac{\Gamma', A, \Delta \ ;\ \Gamma', B, \Delta}{\Gamma', (A \cap B), \Delta}, \qquad (\neg\cap)\ \frac{\Gamma', \neg A, \neg B, \Delta}{\Gamma', \neg(A \cap B), \Delta}$$

Implication rules

$$(\Rightarrow)\ \frac{\Gamma^{'}, \neg A, B, \Delta}{\Gamma^{'}, (A \Rightarrow B), \Delta}, \qquad (\neg \Rightarrow)\ \frac{\Gamma^{'}, A, \Delta \ \ : \ \ \Gamma^{'}, \neg B, \Delta}{\Gamma^{'}, \neg(A \Rightarrow B), \Delta}$$

Negation rule

$$(\neg\neg)\ \frac{\Gamma^{'}, A, \Delta}{\Gamma^{'}, \neg\neg A, \Delta}$$

where $\Gamma^{'} \in \mathcal{F}^{*}, \Delta \in \mathcal{F}'^{*}, A, B \in \mathcal{F}$.

Group 2: Quantifiers Rules

($\exists$)

$$\frac{\Gamma', A(t), \Delta, \exists x A(x)}{\Gamma', \exists x A(x), \Delta}$$

where t is an arbitrary term.

($\forall$)

$$\frac{\Gamma', A(y), \Delta}{\Gamma', \forall x A(x), \Delta}$$

where y is a free individual variable which does not appear in any formula in the conclusion, i.e. in the sequence $\Gamma', \forall x A(x), \Delta$.

The variable y in ($\forall$) is called the **eigenvariable**.

The condition: *where* y *is a free individual variable which does not appear in any formula in the conclusion* is called the **eigenvariable condition**.

All occurrences of y in $A(y)$ of the rule $(\forall)$ are fully indicated.

($\neg\forall$)

$$\frac{\Gamma', \exists x \neg A(x), \Delta}{\Gamma', \neg\forall x A(x), \Delta}$$

($\neg\exists$)

$$\frac{\Gamma', \forall x \neg A(x), \Delta}{\Gamma', \neg\exists x A(x), \Delta}$$

$\Gamma' \in \mathcal{LIT}^*, \Delta \in \mathcal{F}^*, A, B \in \mathcal{F}$.

Note that $A(t), A(y)$ denotes a formula obtained from $A(x)$ by writing t, y, respectively, in place of all occurrences of x in A.