CSE541 EXAMPLE 3: MIDTERM SOLUTIONS

QUESTION 1

Ł semantics for $Ł_{\{\neg,\Rightarrow,\cap,\cup\}}$ is defined as follows

Ł Negation

$\neg$	F	$\perp$	T
	T	$\perp$	F

Ł Disjunction

$\cup$	F	$\perp$	T
F	F	$\perp$	T
$\perp$	$\perp$	$\perp$	T
T	T	T	T

Ł Conjunction

$\cap$	F	$\perp$	T
F	F	F	F
$\perp$	F	$\perp$	$\perp$
T	F	$\perp$	T

Ł-Implication

$\Rightarrow$	F	$\perp$	T
F	T	T	T
$\perp$	$\perp$	T	T
T	F	$\perp$	T

(1) **Use the fact** that $v : VAR \longrightarrow \{F, \perp, T\}$ be such that $v^*((a \cap b) \Rightarrow \neg b) = \perp$ under **Ł** semantics **to evaluate** $v^*(((b \Rightarrow \neg a) \Rightarrow (a \Rightarrow \neg b)) \cup (a \Rightarrow b))$. Use shorthand notation.

(1) Solution : $((a \cap b) \Rightarrow \neg b) = \perp$ in two cases.

C1 $(a \cap b) = \perp$ and $\neg b = F$.

C2 $(a \cap b) = T$ and $\neg b = \perp$.

Case C1: $\neg b = F$, i.e. $b = T$, and hence $(a \cap T) = \perp$ iff $a = \perp$. We get that v is such that $v(a) = \perp$ and $v(b) = T$.

We evaluate: $v^*(((b \Rightarrow \neg a) \Rightarrow (a \Rightarrow \neg b)) \cup (a \Rightarrow b)) = (((T \Rightarrow \neg \perp) \Rightarrow (\perp \Rightarrow \neg T)) \cup (\perp \Rightarrow T)) = ((\perp \Rightarrow \perp) \cup T) = T$.

Case C2: $\neg b = \perp$, i.e. $b = \perp$, and hence $(a \cap \perp) = T$ what is impossible, hence v from case C1 is the only one.

(2) Define the Equivalence of Languages and Prove that in classical semantics $\mathcal{L}_{\{\neg,\Rightarrow\}} \equiv \mathcal{L}_{\{\neg,\Rightarrow,\cup\}}$.

Solution

1. We define the EQUIVALENCE of LANGUAGES as follows:

Given two languages:
$\mathcal{L}_1 = \mathcal{L}_{CON_1}$ and $\mathcal{L}_2 = \mathcal{L}_{CON_2}$, for $CON_1 \neq CON_2$.
We say that they are **logically equivalent**, i.e.

$$\mathcal{L}_1 \equiv \mathcal{L}_2$$

if and only if the following conditions **C1, C2** hold.

C1: For every formula A of $\mathcal{L}_1$, there is a formula B of $\mathcal{L}_2$, such that

$$A \equiv B,$$

C2: For every formula C of $\mathcal{L}_2$, there is a formula D of $\mathcal{L}_1$, such that

$$C \equiv D.$$

2. Proof of equivalence:

C1 holds because any formula of $\mathcal{L}_{\{\neg,\Rightarrow\}}$ is a formula of $\mathcal{L}_{\{\neg,\Rightarrow,\cup\}}$.

C2 holds due to the following definability of connectives equivalence

$$(A \cup B) \equiv (\neg A \Rightarrow B).$$

(3) Prove that the equivalence defining $\cup$ in classical logic does not hold under **Ł** semantics, but nevertheless $\mathcal{L}_{\{\neg,\Rightarrow\}} \equiv \mathcal{L}_{\{\neg,\Rightarrow,\cup\}}$.

Solution $(A \cup B) \not\equiv_{\text{Ł}} (\neg A \Rightarrow B)$ Take $A = B = \perp$. We get $\perp \cup \perp = \perp$ and $\neg \perp \Rightarrow \perp = \perp \Rightarrow \perp = T$.

Proof that $\mathcal{L}_{\{\neg,\Rightarrow\}} \equiv \mathcal{L}_{\{\neg,\Rightarrow,\cup\}}$ holds for **Ł** semantics.

C1: holds because any formula of $\mathcal{L}_{\{\neg,\Rightarrow\}}$ is a formula of $\mathcal{L}_{\{\neg,\Rightarrow,\cup\}}$.

C2: holds because the definability of connectives equivalence

$$(A \cup B) \equiv ((A \Rightarrow B) \Rightarrow B)$$

holds for **Ł**. Easy to check by verification.

Observe that the equivalence $(A \cup B) \equiv (\neg A \Rightarrow B)$ defining $\cup$ in terms of $\neg$ and $\Rightarrow$ is a valuable candidate for (**Ł** semantics definability as the definition of all connectives restricted to T, F is the same as in the classical case. Unfortunately it is not a good one for **Ł** semantics. It does not prove that other definability equivalence does not exist! Observe that the equivalence $(A \cup B) \equiv (A \Rightarrow B) \Rightarrow B)$ provides and alternative proof of **C2** in classical case.

QUESTION 2 Let H be the following proof system:

$$H = (\ \mathcal{L}_{\{\Rightarrow,\neg\}},\ \mathcal{F},\ AX = \{A1, A2, A3, A4\},\quad MP\ )$$

A1 $(A \Rightarrow (B \Rightarrow A))$,

A2 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$

A4 $(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)$

MP (Rule of inference)

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

(1) Justify that H is SOUND under classical semantics.

Solution Axioms **A1-A3** are axioms of a sound system H_2, with the same rule MP. So we need only to check if **A4** is sound, i.e $\models\ (((A \Rightarrow B) \Rightarrow A) \Rightarrow A)$. Assume not, i.e. $((A \Rightarrow B) \Rightarrow A) = T$ and $A = F$. We get $((F \Rightarrow B) \Rightarrow F) = T$. This is impossible, as $(F \Rightarrow B) = T$ for all values of B and $T \Rightarrow F = F$.

(2) Does Deduction Theorem holds for H? Justify shortly your answer.

Solution Axioms **A1-A2** are axioms of system H_1 for which we proved the Deduction Theorem. System H is a (sound) extension of H_1 and hence the Deduction Theorem holds for it as well.

(3) Justify the fact that H is COMPLETE with respect to all classical semantics tautologies.

Solution Axioms **A1-A3** are axioms of system H_2 for which we proved the Completeness Theorem. System H is a (sound) extension of H_2 and hence the Completeness Theorem holds for it as well.

(4) Prove that the system H in NOT COMPLETE under the Lukasiewicz semantics **Ł**.

Solution System H is not sound under **Ł** semantics. For example axiom **A2** is not **Ł** tautology. $A = \perp, B = \perp, C = F$ evaluates it to $\perp$. System that is not sound can't be complete.

(5) All classical tautologies include for example de Morgan Laws

$$(\neg(A \cup B) \Rightarrow (\neg A \cap \neg B)),\quad (\neg(A \cap B) \Rightarrow (\neg A \cup \neg B))$$

Explain what does it mean that they are provable in H.

Solution Obviously $\mathcal{L}_{\{\Rightarrow,\neg\}}$ does not contain connectives $\cup, \cap$ and hence de Morgan Laws as written above are not formulas in our language. But we have proved that

$$\mathcal{L}_{\{\Rightarrow,\neg\}} \equiv \mathcal{L}_{\{\neg,\Rightarrow,\cup,\cap\}}$$

so the corresponding logically equivalent formulas of $\mathcal{L}_{\{\Rightarrow,\neg\}}$ to

$$(\neg(A \cup B) \Rightarrow (\neg A \cap \neg B)), \quad (\neg(A \cap B) \Rightarrow (\neg A \cup \neg B))$$

are also called de Morgan Laws and are provable in H.

(6) Let H' be a proof system obtained from H by adding an additional axiom

A5 $((A \Rightarrow B) \Rightarrow \neg A)$

Is the system H' complete under classical semantics? Justify your answer.

Solution H' is not SOUND (axiom **A5** is not a tautology!) hence can't be complete.

We consider a **sound proof system** (under classical semantics)

$$S = (\ \mathcal{L}_{\{\Rightarrow,\neg\}}, \ \ \mathcal{AX}, \ \ MP\ ),$$

such that the formulas listed below are provable in S.

1. $(A \Rightarrow (B \Rightarrow A))$,

2. $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

3. $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$,

4. $(A \Rightarrow A)$,

5. $(B \Rightarrow \neg\neg B)$,

6. $(\neg A \Rightarrow (A \Rightarrow B))$,

7. $(A \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B)))$,

8. $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$,

9. $((\neg A \Rightarrow A) \Rightarrow A)$.

The following Lemma holds in S

LEMMA For any $A, B, C \in \mathcal{F}$,

(a) $(A \Rightarrow B), (B \Rightarrow C) \vdash_H (A \Rightarrow C)$,

(b) $(A \Rightarrow (B \Rightarrow C)) \vdash_H (B \Rightarrow (A \Rightarrow C))$.

QUESTION 3

Complete the proof sequence (in S)

$$B_1, ..., B_9$$

of

$$((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$$

by providing comments how each step of the proof was obtained.

Solution

$B_1 =$ $(A \Rightarrow B)$
Hypothesis

$B_2 =$ $(\neg\neg A \Rightarrow A)$
Already Proven

$B_3 =$ $(\neg\neg A \Rightarrow B)$
Lemma **a** for $A = \neg\neg A, B = A, C = B$, in B_1, B_2 i.e.

$(\neg\neg A \Rightarrow A), (A \Rightarrow B) \vdash (\neg\neg A \Rightarrow B)$

$B_4 =$ $(B \Rightarrow \neg\neg B)$
Formula 5

$B_5 =$ $(\neg\neg A \Rightarrow \neg\neg B)$
Lemma **a** on B_3, B_4 for $A = \neg\neg A, B = B, C = \neg\neg B$

$(\neg\neg A \Rightarrow B), (B \Rightarrow \neg\neg B) \vdash (\neg\neg A \Rightarrow \neg\neg B)$

$B_6 =$ $((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A))$
ALREADY PROVED

$B_7 =$ $(\neg B \Rightarrow \neg A)$
B_5, B_6 and MP on B_5, B_6

$$\frac{(\neg\neg A \Rightarrow \neg\neg B); ((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A))}{(\neg B \Rightarrow \neg A)}$$

$B_8 =$ $(A \Rightarrow B) \vdash (\neg B \Rightarrow \neg A)$
$B_1 - B_7$

$B_9 = \ ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$
Deduction Theorem on B_8

HERE IS the Main Definition and Main Lemma needed for the PROOF 1 of the Completeness Theorem for the system S.

Main Definition

Let A be a formula and $b_1, b_2, ..., b_n$ be all propositional variables that occur in A. Let v be variable assignment $v : VAR \longrightarrow \{T, F\}$. We define, for any $A, b_1, b_2, ..., b_n$ and v a corresponding formulas A', $B_1, B_2, ..., B_n$ as follows:

$$A' = \begin{cases} A & \text{if } v^*(A) = T \\ \neg A & \text{if } v^*(A) = F \end{cases}$$

$$B_i = \begin{cases} b_i & \text{if } v(b_i) = T \\ \neg b_i & \text{if } v(b_i) = F \end{cases}$$

for $i = 1, 2, ..., n$.

Main Lemma For any formula A and a variable assignment v, if $A^{'}$, B_1 , B_2, ..., B_n are corresponding formulas defined by the definition stated above, then

$$B_1, B_2, ..., B_n \vdash A'.$$

We write $\vdash\ A$ for $\vdash_S A$ as the system S is fixed.

QUESTION 5 We know that the formula

$$A = ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a))$$

is a tautology; i.e. we know that

$$\models\ ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)).$$

Use this information and the method developed in the Proof 1 of Completeness Theorem to show the

$$\vdash\ ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a))$$

Solution This is a shorter solution than one in the Example in the book; it follows directly the proof 1.

We know that A ia a tautology ($\models A$), so $v^*(A) = T$ for all v and $A' = A$ for all v. $A = A(a, b)$, so by the Lemmma $B_1, B_2 \vdash A$, for B_1, B_2 defined accordingly to v, and $v(a), v(b)$.

Step 1: B_2 elimination. $B_2 = b$ if $v(b) = T$ and $B_2 = \neg b$ if $v(b) = F$.
For any v such that $v(b) = T$ we get

$$B_1, b \vdash A$$

and for any v such that $v(b) = F$ we get

$$B_1, \neg b \vdash A.$$

By Deduction Theorem we get

(1) $B_1 \vdash (b \Rightarrow A)$ and $B_1 \vdash (\neg b \Rightarrow A.)$ We have assumed about the proof system S that for ant formulas A, B,

$$\vdash ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$$

so in particular

$$\vdash ((b \Rightarrow A) \Rightarrow ((\neg b \Rightarrow A) \Rightarrow A))$$

and by monotonicity

$$B_1 \vdash ((b \Rightarrow A) \Rightarrow ((\neg b \Rightarrow A) \Rightarrow A)).$$

We apply MP twice to **(1)** and $B_1 \vdash ((b \Rightarrow A) \Rightarrow ((\neg b \Rightarrow A) \Rightarrow A))$ we get that

$$B_1 \vdash A.$$

Step 2: B_1 elimination. $B_1 = a$ if $v(a) = T$ and $B_1 = \neg a$ if $v(a) = F$.
For any v such that $v(a) = T$ we get

$$a \vdash A$$

and for any v such that $v(a) = F$ we get

$$\neg a \vdash A.$$

By Deduction Theorem we get

(2) $\vdash (a \Rightarrow A)$ and $\vdash (\neg a \Rightarrow A.)$ We have assumed about the proof system S that for ant formulas A, B,

$$\vdash ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$$

so in particular

$$\vdash ((a \Rightarrow A) \Rightarrow ((\neg a \Rightarrow A) \Rightarrow A))$$

We apply MP twice to **(2)** and $\vdash ((a \Rightarrow A) \Rightarrow ((\neg a \Rightarrow A) \Rightarrow A))$ we get that

$$\vdash A.$$