Chapter 6: Examples of Propositional Tautologies, Logical Equivalences

Implication

Modus Ponens known to the Stoics (3rd century B.C)

$$\models ((A \cap (A \Rightarrow B)) \Rightarrow B)$$

Detachment

$$\models ((A \cap (A \Leftrightarrow B)) \Rightarrow B)$$

$$\models ((B \cap (A \Leftrightarrow B)) \Rightarrow A)$$

Sufficient Given an implication

$$(A \Rightarrow B),$$

A is called a *sufficient condition* for B to hold.

Necessary Given an implication

$$(A \Rightarrow B),$$

B is called a *necessary condition* for A to hold.

Implication Names

Simple $(A \Rightarrow B)$ is called *a simple implication*.

Converse $(B \Rightarrow A)$ is called *a converse implication* to $(A \Rightarrow B)$.

Opposite $(\neg B \Rightarrow \neg A)$ is called *an opposite implication* to $(A \Rightarrow B)$.

Contrary $(\neg A \Rightarrow \neg B)$ is called *a contrary implication* to $(A \Rightarrow B)$.

Laws of contraposition

$$\models ((A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)),$$

$$\models ((B \Rightarrow A) \Leftrightarrow (\neg A \Rightarrow \neg B)).$$

The laws of contraposition make it possible to replace, in any deductive argument, a sentence of the form $(A \Rightarrow B)$ by $\neg B \Rightarrow \neg A)$, and conversely.

Necessary and sufficient :

We read $(A \Leftrightarrow B)$ as

B is necessary and sufficient for A

because of the following tautology.

$$\models ((A \Leftrightarrow B)) \Leftrightarrow ((A \Rightarrow B) \cap (B \Rightarrow A))).$$

Hypothetical syllogism (Stoics, 3rd century B.C.)

$$\models (((A \Rightarrow B) \cap (B \Rightarrow C)) \Rightarrow (A \Rightarrow C)),$$

$$\models ((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))),$$

$$\models ((B \Rightarrow C) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))).$$

Modus Tollendo Ponens (Stoics, 3rd century B.C.)

$$\models (((A \cup B) \cap \neg A) \Rightarrow B),$$

$$\models (((A \cup B) \cap \neg B) \Rightarrow A)$$

Duns Scotus (12/13 century)

$$\models (\neg A \Rightarrow (A \Rightarrow B))$$

Clavius (16th century)

$$\models ((\neg A \Rightarrow A) \Rightarrow A)$$

Frege (1879, first formulation of the classical propositional logic as a formalized axiomatic system)

$$\models (((A \Rightarrow (B \Rightarrow C)) \cap (A \Rightarrow B)) \Rightarrow (A \Rightarrow C)),$$

$$\models ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$$

Apagogic Proofs : means proofs by *reductio ad absurdum*.

Reductio ad absurdum : to prove A to be true, we assume $\neg A$.

If we get a contradiction, means we have proved A to be true.

$$\models ((\neg A \Rightarrow (B \cap \neg B)) \Rightarrow A)$$

Implication form : we want to prove $(A \Rightarrow B)$ by *reductio ad absurdum*. Correctness of reasoning is based on the following tautologies.

$$\models (((\neg(A \Rightarrow B) \Rightarrow (C \cap \neg C)) \Rightarrow (A \Rightarrow B)),$$

We use the equivalence: $\neg(A \Rightarrow B) \equiv (A \cap \neg B)$ and get

$$\models (((A \cap \neg B) \Rightarrow (C \cap \neg C)) \Rightarrow (A \Rightarrow B)).$$

$$\models (((A \cap \neg B) \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)).$$

$$\models (((A \cap \neg B) \Rightarrow B) \Rightarrow (A \Rightarrow B)).$$

Logical equivalence : For any formulas A, B,

$$A \equiv B \quad iff \quad \models (A \Leftrightarrow B).$$

Laws of contraposition

$$(A \Rightarrow B) \equiv (\neg B \Rightarrow \neg A),$$

$$(B \Rightarrow A) \equiv (\neg A \Rightarrow \neg B),$$

$$(\neg A \Rightarrow B) \equiv (\neg B \Rightarrow A),$$

$$(A \Rightarrow \neg B) \equiv (B \Rightarrow \neg A).$$

Theorem Let B_1 be obtained from A_1 by substitution of a formula B for one or more occurrences of a sub-formula A of A_1, what we denote as

$$B_1 = A_1(A/B).$$

Then the following holds.

$$If \quad A \equiv B, \quad then \quad A_1 \equiv B_1,$$

Definability of Connectives

$$(A \Rightarrow B) \equiv (\neg A \cup B)$$

Transform a formula with implication into a logically equivalent formula without implication.

We transform (via our Theorem) a formula

$$((C \Rightarrow \neg B) \Rightarrow (B \cup C))$$

into its logically equivalent form not containing $\Rightarrow$ as follows.

$$((C \Rightarrow \neg B) \Rightarrow (B \cup C)) \equiv (\neg(C \Rightarrow \neg B) \cup (B \cup C)))$$

$$\equiv (\neg(\neg C \cup B) \cup (B \cup C))).$$

We get

$$((C \Rightarrow \neg B) \Rightarrow (B \cup C)) \equiv (\neg(\neg C \cup B) \cup (B \cup C))).$$