Chapter 6: Definability of Connectives,Equivalence of Languages

Definition of Logical equivalence :

For any formulas A, B,

$$A \equiv B \quad iff \quad \models (A \Leftrightarrow B).$$

Property:

$$A \equiv B \quad iff \quad \models (A \Rightarrow B) \ \ and \ \ \models (B \Rightarrow A).$$

Substitution Theorem Let B_1 be obtained from A_1 by substitution of a formula B for one or more occurrences of a sub-formula A of A_1.

We denote it as

$$B_1 = A_1(A/B).$$

Then the following holds.

$$If \quad A \equiv B, \quad then \quad A_1 \equiv B_1,$$

The next set of equivalences, or corresponding tautologies, deals with what is called *a definability of connectives* in classical semantics.

For example, a tautology

$$\models ((A \Rightarrow B) \Leftrightarrow (\neg A \cup B))$$

makes it possible to define implication in terms of disjunction and negation.

We state it in a form of logical equivalence as follows.

Definability of Implication in terms of negation and disjunction:

$$(A \Rightarrow B) \equiv (\neg A \cup B)$$

We use logical equivalence notion, instead of the tautology notion, as it makes the manipulation of formulas much easier.

Definability of Implication equivalence allows us, by the force of **Substitution Theorem to replace** any formula of the form $(A \Rightarrow B)$ placed anywhere in another formula by a formula $(\neg A \cup B)$.

Hence we transform a given formula containing implication into an logically equivalent formula that does contain implication (but contains negation and disjunction).

Example 1 We transform (via Substitution Theorem) a formula

$$((C \Rightarrow \neg B) \Rightarrow (B \cup C))$$

into its logically equivalent form not containing $\Rightarrow$ as follows.

$$((C \Rightarrow \neg B) \Rightarrow (B \cup C))$$

$$\equiv (\neg(C \Rightarrow \neg B) \cup (B \cup C)))$$

$$\equiv (\neg(\neg C \cup B) \cup (B \cup C))).$$

We get

$$((C \Rightarrow \neg B) \Rightarrow (B \cup C))$$

$$\equiv (\neg(\neg C \cup B) \cup (B \cup C))).$$

It means that that we can, by the Substitution Theorem transform a language

$$\mathcal{L}_1 = \mathcal{L}_{\{\neg, \cap, \Rightarrow\}}$$

into a language

$$\mathcal{L}_2 = \mathcal{L}_{\{\neg, \cap, \cup\}}$$

with all its formulas being logically equivalent.

We write it as the following condition.

C1: for any formula A of $\mathcal{L}_1$, there is a formula B of $\mathcal{L}_2$, such that $A \equiv B$.

Example 2 : Let A be a formula

$$(\neg A \cup (\neg A \cup \neg B))$$

.

We use the definability of implication equivalence to **eliminate disjunction** as follows

$$(\neg A \cup (\neg A \cup \neg B)) \equiv (\neg A \cup (A \Rightarrow \neg B))$$

$$\equiv (A \Rightarrow (A \Rightarrow \neg B)).$$

Observe, that we can't always use the equivalence $(A \Rightarrow B) \equiv (\neg A \cup B)$ to eliminate any disjunction.

For example, we can't use it for a formula

$$A = ((a \cup b) \cap \neg a).$$

In order to be able to transform *any formula* of a language containing **disjunction** (and some other connectives) into a language with negation and implication (and some other connectives), but **without disjunction** we need the following logical equivalence.

Definability of Disjunction in terms of negation and implication:

$$(A \cup B) \equiv (\neg A \Rightarrow B)$$

Example 3 Consider a formula A

$$(a \cup b) \cap \neg a).$$

We transform A into its logically equivalent form not containing $\cup$ as follows.

$$((a \cup b) \cap \neg a) \equiv ((\neg a \Rightarrow b) \cap \neg a).$$

In general, we transform the language $\mathcal{L}_2 = \mathcal{L}_{\{\neg, \cap, \cup\}}$ to the language $\mathcal{L}_1 = \mathcal{L}_{\{\neg, \cap, \Rightarrow\}}$ with all its formulas being logically equivalent.

We write it as the following condition.

C1: for any formula C of $\mathcal{L}_2$, there is a formula D of $\mathcal{L}_1$, such that $C \equiv D$.

The languages $\mathcal{L}_1$ and $\mathcal{L}_2$ for which we the conditions **C1, C2** hold are called **logically equivalent**.

We denote it by

$$\mathcal{L}_1 \equiv \mathcal{L}_2.$$

A general, formal definition goes as follows.

Definition of Equivalence of Languages

Given two languages: $\mathcal{L}_1 = \mathcal{L}_{CON_1}$ and $\mathcal{L}_2 = \mathcal{L}_{CON_2}$, for $CON_1 \neq CON_2$.

We say that they are logically equivalent, i.e.

$$\mathcal{L}_1 \equiv \mathcal{L}_2$$

if and only if the following conditions **C1, C2** hold.

C1: For every formula A of $\mathcal{L}_1$, there is a formula B of $\mathcal{L}_2$, such that

$$A \equiv B,$$

C2: For every formula C of $\mathcal{L}_2$, there is a formula D of $\mathcal{L}_1$, such that

$$C \equiv D.$$

Example 4 To prove the logical equivalence of the languages

$$\mathcal{L}_{\{\neg,\cup\}} \equiv \mathcal{L}_{\{\neg,\Rightarrow\}}$$

we need two definability equivalences:

implication in terms of disjunction and negation,

disjunction in terms of implication and negation, and the **Substitution Theorem.**

Example 5 To prove the logical equivalence of the languages

$$\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}} \equiv \mathcal{L}_{\{\neg,\cap,\cup\}}$$

we need only the definability of implication equivalence.

It proves, by Substitution Theorem that *for any* formula A of

$$\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$$

there is B of $\mathcal{L}_{\{\neg,\cap,\cup\}}$ that equivalent to A, i.e.

$$A \equiv B$$

and condition **C1** holds.

Observe, that any formula A of language

$$\mathcal{L}_{\{\neg,\cap,\cup\}}$$

is also a formula of

$$\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$$

and of course

$$A \equiv A,$$

so **C2** also holds.

The logical equalities below

Definability of Conjunction in terms of implication and negation

$$(A \cap B) \equiv \neg(A \Rightarrow \neg B),$$

Definability of Implication in terms of conjunction and negation

$$(A \Rightarrow B) \equiv \neg(A \cap \neg B),$$

and the **Substitution Theorem** prove that

$$\mathcal{L}_{\{\neg, \cap\}} \equiv \mathcal{L}_{\{\neg, \Rightarrow\}}.$$

Exercise 1

(a) Prove that

$$\mathcal{L}_{\{\cap,\neg\}} \equiv \mathcal{L}_{\{\cup,\neg\}}.$$

(b) Transform a formula $A = \neg(\neg(\neg a \cap \neg b) \cap a)$ of $\mathcal{L}_{\{\cap,\neg\}}$ into a logically equivalent formula B of $\mathcal{L}_{\{\cup,\neg\}}$.

(c) Transform a formula $A = (((\neg a \cup \neg b) \cup a) \cup (a \cup \neg c))$ of $\mathcal{L}_{\{\cup,\neg\}}$ into a formula B of $\mathcal{L}_{\{\cap,\neg\}}$, such that $A \equiv B$.

(d) Prove/disaprove: $\models \neg(\neg(\neg a \cap \neg b) \cap a)$.

(e) Prove/disaprove: $\models (((\neg a \cup \neg b) \cup a) \cup (a \cup \neg c))$.

Solution (a) True due to the Substitution Theorem and two definability of connectives equivalences:

$$(A \cap B) \equiv \neg(\neg A \cup \neg B), \quad (A \cup B) \equiv \neg(\neg A \cap \neg B).$$

Solution (b)

$$\neg(\neg(\neg a \cap \neg b) \cap a)$$

$$\equiv \neg(\neg\neg(\neg\neg a \cup \neg\neg b) \cap a)$$

$$\equiv \neg((a \cup b) \cap a)$$

$$\equiv \neg(\neg(a \cup b) \cup \neg a).$$

The formula B of $\mathcal{L}_{\{\cup,\neg\}}$ equivalent to A is

$$B = \neg(\neg(a \cup b) \cup \neg a).$$

Solution (c)

$$(((\neg a \cup \neg b) \cup a) \cup (a \cup \neg c))$$

$$\equiv((\neg(\neg\neg a \cap \neg\neg b) \cup a) \cup \neg(\neg a \cap \neg\neg c))$$

$$\equiv ((\neg(a \cap b) \cup a) \cup \neg(\neg a \cap c))$$

$$\equiv (\neg(\neg\neg(a \cap b) \cap \neg a) \cup \neg(\neg a \cap c))$$

$$\equiv (\neg((a \cap b) \cap \neg a) \cup \neg(\neg a \cap c))$$

$$\equiv \neg(\neg\neg((a \cap b) \cap \neg a) \cap \neg\neg(\neg a \cap c))$$

$$\equiv\neg(((a \cap b) \cap \neg a) \cap (\neg a \cap c))$$

There are two formulas B of $\mathcal{L}_{\{\cap,\neg\}}$, such that $A \equiv B$.

$$B = B_1 = \neg(\neg\neg((a \cap b) \cap \neg a) \cap \neg\neg(\neg a \cap c)),$$

$$B = B_2 = \neg(((a \cap b) \cap \neg a) \cap (\neg a \cap c)).$$

Solution (d)

$$\not\models \neg(\neg(\neg a \cap \neg b) \cap a)$$

Our formula A is logically equivalent, as proved in **(c)** with the formula
$B = \neg(\neg(a \cup b) \cup \neg a)$.

Consider any truth assignment v, such that
$v(a) = F$, then
$(\neg(a \cup b) \cup T) = T$,
and hence $v^*(B) = F$.

Solution (e)

$$\models (((\neg a \cup \neg b) \cup a) \cup (a \cup \neg c))$$

because it was proved in **(c)** that

$$(((\neg a \cup \neg b) \cup a) \cup (a \cup \neg c))$$

$$\equiv \neg(((a \cap b) \cap \neg a) \cap (\neg a \cap c))$$

and obviously the formula

$$(((a \cap b) \cap \neg a) \cap (\neg a \cap c))$$

is a contradiction.

Hence its negation is a tautology.

Exercise 2 Prove by transformation, using proper logical equivalences that

1.

$$\neg(A \Leftrightarrow B) \equiv ((A \cap \neg B) \cup (\neg A \cap B)),$$

2.

$$((B \cap \neg C) \Rightarrow (\neg A \cup B))$$

$$\equiv ((B \Rightarrow C) \cup (A \Rightarrow B)).$$

Solution 1.

$$\neg(A \Leftrightarrow B)$$

$$\equiv^{def} \neg((A \Rightarrow B) \cap (B \Rightarrow A))$$

$$\equiv^{de\ Morgan} (\neg(A \Rightarrow B) \cup \neg(B \Rightarrow A))$$

$$\equiv^{neg\ impl} ((A \cap \neg B) \cup (B \cap \neg A))$$

$$\equiv^{commut} ((A \cap \neg B) \cup (\neg A \cap B)).$$

Solution 2.

$$((B \cap \neg C) \Rightarrow (\neg A \cup B))$$

$$\equiv^{impl} (\neg(B \cap \neg C) \cup (\neg A \cup B))$$

$$\equiv^{de\ Morgan} ((\neg B \cup \neg\neg C) \cup (\neg A \cup B))$$

$$\equiv^{neg} ((\neg B \cup C) \cup (\neg A \cup B))$$

$$\equiv^{impl} ((B \Rightarrow C) \cup (A \Rightarrow B)).$$