Chapter 8: Hilbert Proof Systems and Deduction Theorem

PROOF OF THE DEDUCTION THEOREM

Hilbert System H_1 :

$$H_1 = (\ \mathcal{L}_{\{\Rightarrow\}},\quad \mathcal{F}\quad \{A1, A2\}\quad MP\ )$$

A1 $(A \Rightarrow (B \Rightarrow A))$,

A2 $((A \Rightarrow (B \Rightarrow C))$
$\Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

MP

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B},$$

DEDUCTION THEOREM (Herbrand, 1930)

For any formulas A, B,

$$if \quad A \vdash B, \quad then \quad \vdash (A \Rightarrow B).$$

We are going to prove now that for our system H_1 is strong enough to prove the Deduction Theorem for it. In fact we prove a more general version of Herbrand theorem. To formulate it we introduce the following notation.

We write

$$\Gamma, A \vdash B$$

for

$$\Gamma \cup \{A\} \vdash B$$

In general we write

$$\Gamma, A_1, A_2, ..., A_n \vdash B$$

for

$$\Gamma \cup \{A_1, A_2, ..., A_n\} \vdash B.$$

Deduction Theorem for H_1 For any subset Γ of the set of formulas $\mathcal{F}$ of H_1 and for any formulas $A, B \in \mathcal{F}$,

$$\Gamma,\ A \vdash_{H_1} B \ \ if\ and\ only\ if\ \ \Gamma \vdash_{H_1} (A \Rightarrow B).$$

In particular,

$$A \vdash_{H_1} B \ \ if\ and\ only\ if\ \ \vdash_{H_1} (A \Rightarrow B).$$

PROOF OF DEDUCTION THEOREM

We use in the proof the symbol $\vdash$ instead of $\vdash_{H_1}$.

Part 1. We first prove:

$$If \quad \Gamma,\ A \vdash B \ \ then \ \ \Gamma \vdash (A \Rightarrow B).$$

Assume that

$$\Gamma,\ A \ \vdash B,$$

i.e. that we have a formal proof

$$B_1, B_2, ..., B_n$$

of B from the set of formulas $\Gamma \cup \{A\}$, we have to show that

$$\Gamma \vdash (A \Rightarrow B).$$

In order to prove that

$$\Gamma \vdash (A \Rightarrow B)$$

follows from $\Gamma,\ A \vdash B$, we prove that a stronger statement, namely that

$$\Gamma \vdash (A \Rightarrow B_i)$$

for any B_i ($1 \leq i \leq n$) in the formal proof $B_1, B_2, ..., B_n$ of B also follows from $\Gamma,\ A \vdash B$.

Hence in particular case, when $i = n$, we will obtain that

$$\Gamma \vdash (A \Rightarrow B)$$

follows from $\Gamma,\ A \vdash B$, and that will end the proof of Part 1.

The proof of Part 1 is conducted by induction on i for $1 \leq i \leq n$.

Step $i = 1$ (base step).

Observe that when $i = 1$, it means that the formal proof

$$B_1, B_2, ..., B_n$$

contains only one element B_1.

By the definition of the formal proof from $\Gamma \cup \{A\}$, we have that

(1) B_1 is a logical axiom, or $B_1 \in \Gamma$, or

(2) $B_1 = A$.

This means that $B_1 \in \{A1, A2\} \cup \Gamma \cup \{A\}$.

Now we have two cases to consider.

Case 1: $B_1 \in \{A1, A2\} \cup \Gamma$**.**

Observe that

$$(B_1 \Rightarrow (A \Rightarrow B_1))$$

is the axiom $A1$ and by assumption

$$B_1 \in \{A1, A2\} \cup \Gamma.$$

We get the required proof of $(A \Rightarrow B_1)$ from Γ by the following application of the Modus Ponens rule

$$(MP) \ \frac{B_1 \ ; \ (B_1 \Rightarrow (A \Rightarrow B_1))}{(A \Rightarrow B_1)}.$$

Case 2: $B_1 = A$

When $B_1 = A$ then to prove $\Gamma \vdash (A \Rightarrow B)$ means to prove

$$\Gamma \vdash (A \Rightarrow A),$$

what holds by the monotonicity of the consequence and the fact that we have shown that

$$\vdash (A \Rightarrow A).$$

The above cases conclude the proof of

$$\Gamma \vdash (A \Rightarrow B_i)$$

for $i = 1$.

INDUCTIVE STEP

Assume that

$$\Gamma \vdash (A \Rightarrow B_k)$$

for all $k < i$,

we will show that using this fact we can conclude that also

$$\Gamma \vdash (A \Rightarrow B_i).$$

Consider a formula B_i in the formal proof $B_1, B_2, ..., B_n$

By the definition of the formal proof we have the following

Case 1 $B_i \in \{A1, A2\} \cup \Gamma \cup \{A\}$or

Case 2: B_i follows by MP from certain B_j, B_m such that $j < m < i$.

We have to consider these cases.

Case 1:

$B_i \in \{A1, A2\} \cup \Gamma \cup \{A\}$.

The proof of $(A \Rightarrow B_i)$ from Γ in this case is obtained from the proof of the **Step** $i = 1$ by replacement B_1 by B_i and will be omitted here as a straightforward repetition.

Case 2:

B_i is a conclusion of MP.

If B_i is a conclusion of MP, then we must have two formulas B_j, B_m in the formal proof

$B_1, B_2, ..., B_n$ such that $j < m < i$ and

$$(MP)\ \frac{B_j\ ;\ B_m}{B_i}.$$

By the inductive assumption,

the formulas B_j, B_m are such that

$$\Gamma \vdash (A \Rightarrow B_j)$$

and

$$\Gamma \vdash (A \Rightarrow B_m).$$

Moreover, by the definition of Modus Ponens rule, the formula B_m has to have a form

$$(B_j \Rightarrow B_i),$$

i.e.

$$B_m = (B_j \Rightarrow B_i),$$

and the inductive assumption can be re-written as follows.

$$\Gamma \vdash (A \Rightarrow (B_j \Rightarrow B_i))$$

, for $j < i$.

Observe now that the formula

$$((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))$$

is a substitution of the axiom A2 and hence **has a proof** in our system.

By the monotonicity of the consequence, it also has a proof from the set Γ, i.e.

We know that

$$\Gamma \vdash ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))).$$

Applying the rule MP i.e. performing the following

$$\frac{(A \Rightarrow (B_j \Rightarrow B_i)) \ ; \ ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))}{((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))}$$

we get that also

$$\Gamma \vdash((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)).$$

Applying again the rule MP i.e. performing the following

$$\frac{(A \Rightarrow B_j) \; ; \; ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))}{(A \Rightarrow B_i)})$$

we get that

$$\Gamma \vdash (A \Rightarrow B_i)$$

what ends the proof of the inductive step.

By the mathematical induction principle, we hence have proved that

$$\Gamma \vdash (A \Rightarrow B_i)$$

for all i such that $1 \leq i \leq n$.

In particular it is true for $i = n$, what means for $B_n = B$.

This ends the proof of the fact that

if $\Gamma, A \vdash B$, then $\Gamma \vdash (A \Rightarrow B)$.

The proof of the inverse implication:

$$If \Gamma \vdash_{H_1} (A \Rightarrow B), \quad then \quad \Gamma,\ A \vdash_{H_1} B$$

is straightforward and goes as follows.

Assume that

$\Gamma \vdash (A \Rightarrow B)$.

By the monotonicity of the consequence we have also that

$\Gamma, A \ \vdash (A \Rightarrow B)$.

Obviously

$\Gamma, A \ \vdash A$.

Applying Modus Ponens to the above, we get the proof of B from $\{\Gamma, A\}$ i.e.

we have proved that

$$\Gamma, A \ \vdash B.$$

THIS ENDS the proof of the deduction theorem for any set $\Gamma \subseteq \mathcal{F}$ and any formulas $A, B \in \mathcal{F}$.

The particular case of the theorem is obtained from the above by assuming that the set Γ is empty.