Chapter 8: Hilbert Systems, Deduction Theorem

Introduction

Hilbert Systems The Hilbert proof systems are based on a language with implication and contain a Modus Ponens rule as a rule of inference.

Modus Ponens is the oldest of all known rules of inference as it was already known to the Stoics (3rd century B.C.).

It is also considered as the most "natural" to our intuitive thinking and the proof systems containing it as the inference rule play a special role in logic.

Hilbert System H_1 :

$$H_1 = (\ \mathcal{L}_{\{\Rightarrow\}},\quad \mathcal{F}\quad \{A1, A2\}\quad MP\ )$$

A1 $(A \Rightarrow (B \Rightarrow A))$,

A2 $((A \Rightarrow (B \Rightarrow C))$
$\Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

MP

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B},$$

Finding formal proofs in this system requires some ingenuity.

The formal proof of $(A \Rightarrow A)$ in H_1 is a sequence

$$B_1,\ B_2,\ B_2,\ B_2, B_5$$

as defined below.

$B_1 = ((A \Rightarrow ((A \Rightarrow A) \Rightarrow A)) \Rightarrow ((A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A))),$
axiom A2 for $A = A$, $B = (A \Rightarrow A)$, and $C = A$

$B_2 = (A \Rightarrow ((A \Rightarrow A) \Rightarrow A)),$
axiom A1 for $A = A$, $B = (A \Rightarrow A)$

$B_3 = ((A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A))),$
MP application to B_1 and B_2

$B_4 = (A \Rightarrow (A \Rightarrow A)),$
axiom A1 for $A = A$, $B = A$

$B_5 = (A \Rightarrow A)$
MP application to B_3 and B_4

A general procedure for searching for proofs in a proof system S can be stated is as follows.

Given an expression B of the system S. If it has a proof, it must be conclusion of the inference rule. Let's say it is a rule r.

We find its premisses, with B being the conclusion, i.e. we evaluate $r^{-1}(B)$.

If all premisses are axioms, the proof is found.

Otherwise we repeat the procedure for any non-axiom premiss.

Search for proof by the means of MP The MP rule says: given two formulas A and $(A \Rightarrow B)$ we can conclude a formula B.

Assume now that we have a formula B and want to find its proof.

If B is an axiom, we have the proof: the formula itself.

If it is not an axiom, it had to be obtained by the application of the Modus Ponens rule, to certain two formulas A and $(A \Rightarrow B)$.

But there is infinitely many of formulas A and $(A \Rightarrow B)$. I.e. for any B, the inverse image of B under the rule MP, $MP^{-1}(B)$ is countably infinite.

The proof system H_1 is not syntactically decidable.

Semantic Link 1 System H_1 is sound under classical semantics and is not sound under **Ł** semantics.

Soundness Theorem of H_1 For any $A \in \mathcal{F}$ of H_1,

$$If \quad \vdash_{H_1} A, \quad then \quad \models A.$$

Semantic Link 2 The system H_1 **is not complete** under classical semantics.

Not all classical tautologies have a proof in H_1.

$\models (\neg\neg A \Rightarrow A)$ and $\not\vdash_{H_1} (\neg\neg A \Rightarrow A)$.

Exercise: show that

$$(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C).$$

We construct a formal proof

$$B_1, B_2, ...B_7,$$

as follows.

$B_1 = (B \Rightarrow C)$, hypothesis

$B_2 = (A \Rightarrow B)$, hypothesis

$B_3 = ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,
axiom A2

$B_4 = ((B \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C)))$,
axiom A1 for $A = (B \Rightarrow C), B = A$

$B_5 = ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$,
B_1 and B_4 and MP

$B_6 = ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$,
B_3 and B_5 and MP

$B_7 = (A \Rightarrow C)$
B_1, B_6 and MP

In mathematical arguments, one often a statement B on the assumption (hypothesis) of some other statement A and then concludes that we have proved the implication "if A, then B".

This reasoning is justified by the following theorem, called a Deduction Theorem.

Notation: $\Gamma, A \vdash B$ for $\Gamma \cup \{A\} \vdash B$,

In general: $\Gamma, A_1, A_2, ..., A_n \vdash B$

for $\Gamma \cup \{A_1, A_2, ..., A_n\} \vdash B$.

Deduction Theorem for H_1

$$\Gamma,\ A \vdash_{H_1} B \quad iff \ \Gamma \vdash_{H_1} (A \Rightarrow B).$$

In particular ,

$$A \vdash_{H_1} B \quad iff \vdash_{H_1} (A \Rightarrow B).$$

Lemma :

$$\textbf{(a)} \quad (A \Rightarrow B), (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C),$$

$$\textbf{(b)} \quad (A \Rightarrow (B \Rightarrow C)), B \vdash_{H_1} (A \Rightarrow C).$$

First we construct a formal proof

$$B_1, B_2, B_3, B_4, B_5$$

as follows.

$B_1 = (A \Rightarrow B)$, hypothesis	$B_2 = (B \Rightarrow C)$, hypothesis	$B_3 = A$ hypothesis
$B_4 = B$ B_1, B_3 and MP	$B_5 = C$ B_2, B_4 and MP	

Thus we proved :

$$(A \Rightarrow B), (B \Rightarrow C), A \vdash_{H_1} A.$$

By Deduction Theorem, we get

$$(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C).$$

Hilbert System $H_2 = (\ \mathcal{L}_{\{\Rightarrow,\neg\}},\ \ A1, A2, A3,\ \ MP\ )$

A1 (Law of simplification)

$(A \Rightarrow (B \Rightarrow A))$,

A2 (Frege's Law)

$((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$

MP (Rule of inference)

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B},$$

and A, B, C are any formulas of the propositional language $\mathcal{L}_{\{\Rightarrow,\neg\}}$.

We write :

$$\vdash_{H_2} A$$

to denote that a formula A has a formal proof in H_2 (from the set of logical axioms $A1, A2, A3$), and

$$\Gamma \vdash_{H_2} A$$

to denote that a formula A has a formal proof in H_2 from a set of formulas Γ (and the set of logical axioms $A1, A2, A3$.

Observe that system H_2 was obtained by by adding axiom A_3 to the system H_1. Hence the Deduction Theorem holds for system H_2 as well. I.e the following theorem holds.

Deduction Theorem for H_2: For any subset Γ of the set of formulas $\mathcal{F}$ of H_2 and for any formulas $A, B \in \mathcal{F}$,

$\Gamma,\ A \vdash_{H_2} B \ \ if\ and\ only\ if \ \ \Gamma \vdash_{H_2} (A \Rightarrow B).$

In particular,

$$A \vdash_{H_2} B \ \ if\ and\ only\ if \ \ \vdash_{H_2} (A \Rightarrow B).$$

Soundness Theorem for H_2:

For every formula $A \in \mathcal{F}$,

$$if \ \vdash_{H_2} A, \ then \ \models A.$$

The soundness theorem proves that our prove system "produces" only tautologies.

We show, as the next step, that in our proof system all tautologies can be proven.

This is called a completeness part of the *completeness theorem for classical logic.*