Sequents

A *sequent* is an expression of the form $\Gamma \vdash \phi$, where Γ is a finite set of formulas and ϕ is a formula.

A sequent $\Gamma \vdash \phi$ expresses, in the context of a given proof system, that a formulas ϕ can be derived from assumptions (or premises) represented by the formulas in Γ, or that ϕ depends on a set of assumptions Γ.

There are also proof systems that are formulated directly in terms of sequents (so-called *sequent calculi*), though in those cases one usually considers more general sequents $\Gamma \vdash \Delta$, where Γ and Δ are finite sets (sometimes sequences or multisets) of formulas. (The set Γ is called the *antecedent*; the set Δ, the *succedent* of the sequent.)

A sequent $\Gamma \vdash \phi$ is said to be *valid* if for each truth valuation under which all formulas in Γ are true, the formula ϕ is also true. It is *falsifiable* if there is a truth assignment that makes Γ true, but ϕ false.

A proof system is said to be *sound* if every provable sequent is valid, and *complete* if all valid sequents are provable. If a proof system is sound and complete, provability coincides with logical consequence.

Natural Deduction

We next present a sequent-style formulation of natural deduction.

This proof system contains as axioms all sequents $\Gamma \vdash \phi$, where ϕ is an element of Γ.

The system also contains inference rules for each logical symbol, which can be divided into *introduction* and *elimination rules*. In describing these rules, we write Γ, ϕ to denote the set of formulas $\Gamma \cup \{\phi\}$.

Conjunction

We have one introduction rule,

$$\frac{\Gamma \vdash \phi \qquad \Delta \vdash \psi}{\Gamma, \Delta \vdash \phi \wedge \psi},$$

and two elimination rules,

$$\frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \phi}$$

and

$$\frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \psi}.$$

Natural Deduction - Disjunction

Disjunction

We have two introduction rules,

$$\frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \vee \psi}$$

and

$$\frac{\Gamma \vdash \psi}{\Gamma \vdash \phi \vee \psi},$$

and one elimination rule,

$$\frac{\Gamma \vdash \phi \vee \psi \qquad \Delta, \phi \vdash \gamma \qquad \Theta, \psi \vdash \gamma}{\Gamma, \Delta, \Theta \vdash \gamma}.$$

Natural Deduction - Implication

Implication

The introduction rule is

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \to \psi},$$

the elimination rule,

$$\frac{\Gamma \vdash \phi \to \psi \qquad \Delta \vdash \phi}{\Gamma, \Delta \vdash \psi}.$$

Negation

The introduction rule is

$$\frac{\Gamma, \phi \vdash \bot}{\Gamma \vdash \neg\phi},$$

the elimination rule,

$$\frac{\Gamma \vdash \neg\phi \qquad \Delta \vdash \phi}{\Gamma, \Delta \vdash \bot}.$$

Note that the negation rules may be considered special cases of the implication rules, if one interprets $\neg\phi$ as an abbreviation of $\phi \to \bot$.

Natural Deduction - Contradiction

Proof by Contradiction

$$\frac{\Gamma, \neg\phi \vdash \bot}{\Gamma \vdash \phi}$$

A weaker (intuitionistic) version of a contradiction rule is

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \phi},$$

which (for classical logic) is supplemented by

Double Negation Elimination

$$\frac{\Gamma \vdash \neg\neg\phi}{\Gamma \vdash \phi}.$$

Natural Deduction - Derived Rules

Useful derived inference rules include:

Modus Tollens

$$\frac{\Gamma \vdash \phi \rightarrow \psi \qquad \Delta \vdash \neg\psi}{\Gamma, \Delta \vdash \neg\phi}$$

Law of the Excluded Middle

$$\frac{}{\vdash \phi \vee \neg\phi}$$

Derivations and Proofs

By a *derivation tree* we mean a labeled binary tree, such that each non-leaf node is labeled by a sequent that matches the conclusion of an inference rule and its children are labeled by the corresponding premises of the inference rule.

A derivation tree is called a *proof tree* if all its leaves are (labeled by) axioms.

Proofs can also be represented as sequences. Formally, a *proof* of $\Gamma \vdash \phi$ is a sequence of sequents, $S_1, \ldots, S_n$, where $S_n = \Gamma \vdash \phi$ and each sequent S_i, $1 \leq i \leq n$, is either an axiom or else is the conclusion of an inference, all premises of which are sequents S_j with $j < i$.

We say that a sequent $\Gamma \vdash \phi$ is *provable* if there is a proof (tree) for it.

Soundness

The following lemma can be easily established via inspection of the different rules of the natural deduction claculus:

Lemma

> A truth assignment falsifies the conclusion of a natural deduction rule if, and only if, it falsifies at least one of the premises.

As a corollary we obtain the:

Theorem (Soundness of Natural Deduction)

> If a sequent $\Gamma \vdash \phi$ is provable then ϕ is a logical consequence of Γ.

The theorem can be proved by induction on (the length or size of) proofs, using the above lemma.

A derivation tree is also called a *counterexample* if some leaf contains no logical connectives, but is not an axiom.

Completeness

The second important property of the natural deduction calculus is its completeness.

Theorem (Completeness)

> If ϕ is a logical consequence of Γ, then the sequent $\Gamma \vdash \phi$ is provable.

The key idea for the proof of the Completeness Theorem is that one can (repeatedly) use the law of the excluded middle to construct a proof with 2^n subproofs, where n is the number of different variables occurring in $\Gamma \vdash \phi$. Each subproof contains assumptions encoding one of the 2^n possible truth assignments for these variables, and demonstrates the validity of the goal $\Gamma \vdash \phi$ for the encoded truth valuation. Details are given in the textbook.