

Well-founded orderings

An (strict partial) *order* (or *ordering*) on a set S is an irreflexive and transitive binary relation on S .

In other words, a binary relation \succ on S is a strict order if it satisfies the following two properties, for all elements x , y , and z of S :

1. $x \not\succeq x$ and
2. whenever $x \succ y$ and $y \succ z$, then $x \succ z$.

Examples of strict orderings are the less-than and the greater-than relation on the integers.

(The less-than-or-equal-to relation is a *non-strict order*, that is, it is a reflexive, transitive, and antisymmetric binary relation.)

An order \succ is said to be *total* on a set S if, for all elements x and y of S , we have $x \succ y$, or $y \succ x$, or $x = y$.

Finally, an order \succ is called *well-founded* if there is no infinite descending chain $x_1 \succ x_2 \succ \dots$.

The less-than and greater-than are both total, but are not well-founded on the integers.

Noetherian Induction

Well-founded orderings are important because they allow one to generalize mathematical induction in the following way.

Principle of Noetherian Induction

Let \succ be a well-founded ordering on a set S and P be a property defined on elements of S .

If for all elements x in S , the assumption that $P(y)$ is true for all y with $x \succ y$, implies that $P(x)$ is true, then $P(x)$ is true for all x in S .

This principle can be used to (a) prove properties on well-founded sets and (b) define properties (or functions) on well-founded sets.

Lexicographic Orderings

If \succ_1 is a strict ordering on a set S_1 and \succ_2 a strict ordering on a set S_2 , then their *lexicographic combination* \succ_{lex} is defined by:

$(x, y) \succ_{lex} (x', y')$ if and only if either $x \succ_1 x'$ or else $x = x'$ and $y \succ_2 y'$.

This lexicographic combination is a strict ordering on the set $S_1 \times S_2$.

Theorem.

If \succ_1 and \succ_2 are well-founded orderings, so is their lexicographic combination \succ_{lex} .

More generally lexicographic orderings can be defined on triples, quadruples, or in fact n -tuples of elements (for any fixed positive integer n).

Multisets

Informally a *multiset* is an unordered collection of elements with possible repeated occurrences.

For example, $\{2, 3, 4, 3, 5, 2, 3\}$ and $\{2, 2, 3, 3, 3, 4, 5\}$ denote the same multiset of integers, which is different from the multiset $\{2, 3, 4, 5\}$.

Formally a multiset of elements from a set S is a mapping M from S to the natural numbers. The number $M(x)$ indicates how often the element x occurs in M .

We say that x is an *element of a multiset* M if $M(x) > 0$.

A multiset M is said to be *finite* if $M(x) = 0$ for all but a finite number of elements $x \in S$.

The union and intersection of two multisets M and M' are defined by the following identities:

1. $(M \cup M')(x) = M(x) + M'(x)$ and
2. $(M \cap M')(x) = \min\{M(x), M'(x)\}$.

Multiset orderings

Let \succ be a strict partial ordering on a set S . We define a corresponding ordering on finite multisets over S as follows:

$M \succ_{mul} M'$ if (i) $M \neq M'$ and (ii) for all elements x in S , such that $M'(x) > M(x)$, there exists an element $y \in S$, such that $y \succ x$ and $M(y) > M'(y)$.

(The symbol $>$ denotes the greater-than relation on the integers.)

We also call \succ_{mul} the *multiset extension* of \succ .

Intuitively, $M \succ M'$ if M' can be obtained from M by replacing some elements by finitely many (possibly zero) smaller elements.

Lemma.

The multiset extension of a strict partial order is also a strict partial order.

The multiset extension of a total order on a set S is total on finite multisets over S .

Multiset Orderings – Well-foundedness

We are interested in the following property of multiset orderings.

Theorem

The multiset extension of a well-founded ordering on a set S is well-founded on finite multisets over S .

Sketch of proof.

The theorem can be proved by contradiction, using König's Infinity Lemma.

Suppose there is an infinite descending chain of finite multisets $M_1 \succ M_2 \succ \dots$.

We construct a corresponding finitely branching tree as follows.

The root of the tree is labeled by the symbol \top and has children labeled by the elements of M_1 .

Now M_2 can be obtained from M_1 by replacing some elements with finitely many smaller elements. We represent each replacement by adding corresponding children labeled with smaller elements to a node.

If an element is deleted (replaced by zero elements), we add a child labeled by the symbol \perp .

The infinite descending chain of multisets thus induces a finitely branching, infinite tree.

By the Infinity Lemma there must be an infinite branch. But this would imply that there exists an infinite descending chain of elements of S , which contradicts the assumption that S is well-founded.

Note. For a discussion of the Infinity Lemma, and other applications, see D.E. Knuth, *The Art of Computer Programming*, vol. 1, pp. 382–386.