GL, GI: FEW PROBLEMS

QUESTION 1 Let **GL** be the Gentzen style proof system for classical logic. Prove, by constructing a proper decomposition tree that

(1) $\vdash_{\mathbf{GL}} ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$.

Solution: By definition we have that

$$\vdash_{\mathbf{GL}} ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \quad if\ and\ only\ if$$

$$\vdash_{\mathbf{GL}} \longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)).$$

$$\mathbf{T}_{\longrightarrow A}$$

$$\begin{array}{c}
\longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \\
\mid (\rightarrow\Rightarrow) \\
(\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a) \\
\mid (\rightarrow\Rightarrow) \\
\neg b, (\neg a \Rightarrow b) \longrightarrow a \\
\mid (\rightarrow \neg) \\
(\neg a \Rightarrow b) \longrightarrow b, a \\
\bigwedge (\Rightarrow\longrightarrow)
\end{array}$$

$$\begin{array}{ccc}
\longrightarrow \;\; \neg b, b, a & \qquad\qquad & b \longrightarrow b, a \\
\mid (\rightarrow \neg) & & axiom \\
b \;\; \longrightarrow \;\; b, a & & \\
axiom & &
\end{array}$$

All leaves of the tree are axioms, hence we have found the proof of A in **GL**.

(2) Let **GL** be the Gentzen style proof system for classical propositional logic. Prove, by constructing proper decomposition trees that

$$\nvdash_{\mathbf{GL}}((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)).$$

DEFINITION $\vdash_{\mathbf{GL}} A$ iff There is a tree $\mathbf{T}_{\rightarrow A}$ such that all its leaves are axioms.

Hence $\nvdash_{\mathbf{GL\ A}}$ iff any decomposition tree $\mathbf{T}_{\rightarrow A}$ has an non-axiom leaf.

Solution: Observe that for any formula A, its decomposition tree $\mathbf{T}_{\rightarrow A}$ in **GL** is not unique. Hence when constructing decomposition trees we have to cover all possible cases.

We construct the decomposition tree for $\longrightarrow A$ as follows.

$$\mathbf{T_{1 \rightarrow A}}$$

$$\longrightarrow ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\rightarrow \Rightarrow)$$

$$(one\ choice)$$

$$(a \Rightarrow b) \ \longrightarrow \ (\neg b \Rightarrow a)$$

$$\mid (\rightarrow \Rightarrow)$$

$$(first\ of\ two\ choices)$$

$$\neg b, (a \Rightarrow b) \ \longrightarrow \ a$$

$$\mid (\neg \rightarrow)$$

$$(one\ choice)$$

$$(a \Rightarrow b) \ \longrightarrow \ b, a$$

$$\bigwedge (\Rightarrow \longrightarrow)$$

$$(one\ choice)$$

$$\longrightarrow \ a, b, a \qquad\qquad b \longrightarrow b, a$$

$$non - axiom \qquad\qquad axiom$$

The tree contains a non- axiom leaf $\longrightarrow \ a, b, a$, hence it is not a proof in **GL**. We have only one more tree to construct. Here it is.

$\mathbf{T_{2 \rightarrow A}}$

$$\longrightarrow ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(one\ choice)$$

$$(a \Rightarrow b) \ \longrightarrow \ (\neg b \Rightarrow a)$$

$$\bigwedge(\Rightarrow\longrightarrow)$$

$$(second\ of\ two\ choices)$$

$\longrightarrow \ (\neg b \Rightarrow a), a$	$b \longrightarrow \ (\neg b \Rightarrow a)$
$(\longrightarrow\Rightarrow)$	$\mid (\rightarrow\Rightarrow)$
$(one\ choice)$	$(one\ choice)$
$\neg b \longrightarrow \ a, a$	$b, \neg b \longrightarrow \ a$
$\mid (\neg \rightarrow)$	$\mid (\neg \rightarrow)$
$(one\ choice)$	$(one\ choice)$
$\longrightarrow \ a, a, b$	$b \ \longrightarrow \ a, b$
$non - axiom$	$axiom$

All possible trees end with an non-axiom leave whet proves that

$$\not\vdash_{\mathbf{GL}}((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)).$$

QUESTION 2 Does the tree below constitute a proof in **GL**? Justify your answer.

$$\mathbf{T}_{\longrightarrow A}$$

$$\longrightarrow \neg\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$
$$\mid (\to \neg)$$
$$\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow$$
$$\mid (\neg \to)$$
$$\longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$
$$\mid (\to\Rightarrow)$$
$$(\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$
$$\mid (\to\Rightarrow)$$
$$(\neg a \Rightarrow b), \neg b \longrightarrow a$$
$$\mid (\neg \to)$$
$$(\neg a \Rightarrow b) \longrightarrow b, a$$
$$\bigwedge (\Rightarrow\longrightarrow)$$

$\longrightarrow \neg a, b, a$	$b \longrightarrow b, a$
$\mid (\to \neg)$	*axiom*
$a \longrightarrow b, a$	
axiom	

Q2 Solution: The tree is a not a proof in **GL** because a rule corresponding to the decomposition step below **does not exists** in it.

$$(\neg a \Rightarrow b), \neg b \longrightarrow a$$

$$\mid (\neg \rightarrow)$$

$$(\neg a \Rightarrow b) \longrightarrow b, a$$

The tree is a proof is some system **GL1** that has all the rules of **GL** except its $(\neg \rightarrow)$ rule:

$$(\neg \rightarrow) \ \frac{\Gamma^{'}, \Gamma \ \longrightarrow \ \Delta, A, \Delta^{'}}{\Gamma^{'}, \neg A, \Gamma \ \longrightarrow \ \Delta, \Delta^{'}},$$

This rule has to be replaced by the rule:

$$(\neg \rightarrow)_1 \ \frac{\Gamma, \Gamma^{'} \ \longrightarrow \ \Delta, A, \Delta^{'}}{\Gamma, \neg A, \Gamma^{'} \ \longrightarrow \ \Delta, \Delta^{'}}.$$

QUESTION 3 Let **GL** be the Gentzen style proof system for classical logic defined in chapter 11. Prove, by constructing a counter-model defined by a proper decomposition tree that

$$\not\models ((a \Rightarrow (\neg b \cap a)) \Rightarrow (\neg b \Rightarrow (a \cup b))).$$

Solution next page

$$\mathbf{T}_{\longrightarrow A}$$

$$\longrightarrow ((a \Rightarrow (\neg b \cap a)) \Rightarrow (\neg b \Rightarrow (a \cup b)))$$

$$\mid (\longrightarrow\Rightarrow)$$

$$(a \Rightarrow (\neg b \cap a)) \longrightarrow (\neg b \Rightarrow (a \cup b))$$

$$\mid (\longrightarrow\Rightarrow))$$

$$one\ of\ two\ choices$$

$$\neg b, (a \Rightarrow (\neg b \cap a)) \longrightarrow (a \cup b))$$

$$\mid (\longrightarrow \cup)$$

$$one\ of\ two\ choices$$

$$\neg b, (a \Rightarrow (\neg b \cap a)) \longrightarrow a, b$$

$$\mid (\neg \longrightarrow)$$

$$(a \Rightarrow (\neg b \cap a)) \longrightarrow b, a, b$$

$$\bigwedge (\Rightarrow\longrightarrow)$$

$$\longrightarrow \neg a, b, a, b \qquad\qquad \longrightarrow (\neg b \cap a), b, a$$

$$\mid (\longrightarrow \neg) \qquad\qquad \bigwedge (\longrightarrow \cap)$$

$$a \longrightarrow b, a, b$$

$$axiom \qquad \longrightarrow \neg b, b, a, b \qquad \longrightarrow a, b, a, b$$

$$\mid (\longrightarrow \neg) \qquad non-axiom$$

$$b \longrightarrow b, a, b$$

$$axiom$$

The counter-model model determined by the non-axiom leaf

$$\longrightarrow a, b, a, b$$

is any truth assignment that evaluates it to F.

Observe that (we use a shorthand notation)

$\longrightarrow a, b, a, b$ represents semantically $T \longrightarrow a, b, a, b$ and hence

$\longrightarrow a, b, a, b = F$ iff $T \longrightarrow a, b, a, b = F$,

what happens only if

$$T \Rightarrow a \cup b \cup a \cup b = F,$$

i.e when $a = F$ and $b = F$.

QUESTION 4 Prove the COMPLETENESS theorem for GL. Assume that the Soundness has been already proved and the Decompositions Trees are already defined.

Solution

Formula Completeness: For any $A \in \mathcal{F}$,

$$\models A \quad iff \quad \vdash_{GL} \rightarrow A$$

Soundness part: for any $A \in \mathcal{F}$,

$$If \vdash_{GL} \rightarrow A, \quad then \quad \models A$$

Completeness part : for any $A \in \mathcal{F}$,

$$If \quad \models A, \; then \vdash_{GL} \rightarrow A$$

We prove the logically equivalent form of the Completeness part: for any $A \in \mathcal{F}$,

$$If \not\vdash_{GL} \rightarrow A \ then \ \not\models A,$$

proof Assume $\not\vdash_{GL} \rightarrow A$, i.e. $\rightarrow A$ does not have a proof in GL. Let $\mathcal{T}_A$ be a set of all decomposition trees of $\rightarrow A$. As $\not\vdash_{GL} \rightarrow A$, each $T \in \mathcal{T}_A$ has a non-axiom leaf. We choose an arbitrary $T_A \in \mathcal{T}_A$. Let $\Gamma' \rightarrow \Delta', \Gamma', \Delta' \in VAR^*$ be an non-axiom leaf of T_A, i.e. $\{\Gamma'\} \cap \{\Delta'\} = \emptyset$.

The non-axiom leaf

$$\Gamma' \rightarrow \Delta'$$

defines a truth assignment $v : VAR \rightarrow \{T, F\}$ which falsifies A, as follows:

$$v(a) = \begin{cases} T & \text{if a appears in } \Gamma' \\ F & \text{if a appears in } \Delta' \end{cases}$$

This proves, by strong soundness of the rules of inference of GL that $\not\models A$.

QUESTION 5 Let **LI** be the Gentzen system for intuitionistic logic as defined in chapter 12.
Show that

$$\vdash_{\mathbf{LI}} \neg\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)).$$

Solution: Observe that

$$\vdash_{\mathbf{LI}} \neg\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

if and only if

$$\vdash_{\mathbf{LI}} \longrightarrow \neg\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \quad .$$

Consider the following decomposition tree $\mathbf{T}_{\rightarrow A}$ of $\rightarrow \neg\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$ in **LI**.

$$\mathbf{T}_{\longrightarrow A}$$

$$\longrightarrow \neg\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\to \neg)$$

$$\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow$$

$$\mid (contr \to)$$

$$\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)), \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow$$

$$\mid (\neg \to)$$

$$\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\to\Rightarrow)$$

$$(\neg a \Rightarrow b), \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow (\neg b \Rightarrow a)$$

$$\mid (\to\Rightarrow)$$

$$\neg b, (\neg a \Rightarrow b), \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow a$$

$$\mid (exch \to)$$

$$(\neg a \Rightarrow b), \neg b, \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow a$$

$$\bigwedge(\Rightarrow\longrightarrow)$$

$$Left \quad premiss$$

$$\neg b, \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow \neg a$$

$$\mid (\rightarrow \neg)$$

$$a, \neg b, \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow$$

$$\mid (exch \rightarrow)$$

$$a, \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)), \neg b \longrightarrow$$

$$\mid (exch \rightarrow)$$

$$\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)), a, \neg b \longrightarrow$$

$$\mid (\neg \rightarrow)$$

$$a, \neg b \longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\rightarrow \Rightarrow)$$

$$(\neg a \Rightarrow b), a, \neg b \longrightarrow (\neg b \Rightarrow a)$$

$$\mid (\rightarrow \Rightarrow)$$

$$\neg b, (\neg a \Rightarrow b), a, \neg b \longrightarrow a$$

$$axiom$$

$$Right \quad premiss$$

$$b, \neg b, \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow a$$

$$\mid (exch \rightarrow)$$

$$\neg b, b, \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow a$$

$$\mid (\rightarrow weak), (\neg \rightarrow)$$

$$b, \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow b$$

$$axiom$$

All leaves of $\mathbf{T}_{\to A}$ are axioms, we have hence found a proof.

QUESTION 6 We know that the formulas below **are not** Intuitionistic Tautologies. Verify whether **H** semantics (chapter 5) provides a counter-model for them.

$$((a \Rightarrow b) \Rightarrow (\neg a \cup b))$$

$$((\neg a \Rightarrow \neg b) \Rightarrow (b \Rightarrow a))$$

Solution

First Formula:

We evaluate:$((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = \perp$ iff $(a \Rightarrow b) = T$ and $(\neg a \cup b) = \perp$. Observe that $(\neg a \cup b) = \perp$ in 3 cases, two of which for $\neg a = \perp$ are impossible. We have hence only one case to consider: $\neg a = F, b = \perp$, i.e. $a = \perp$ or $a = T$ and $b = \perp$. Both of them provide a counter-model.

Solution for second formula

$$((\neg a \Rightarrow \neg b) \Rightarrow (b \Rightarrow a)) = \perp$$

if and only if

$$(\neg a \Rightarrow \neg b) = T$$

and

$$(b \Rightarrow a) = \perp .$$

The case $(b \Rightarrow a) = \perp$ holds iff $b = T$ and $a = \perp$. In this case $(\neg a \Rightarrow \neg b) = (\neg \perp \Rightarrow \neg T) = F \Rightarrow F = T$. We have a counter-model.

QUESTION 7 Show that

$$\vdash_{\mathbf{LI}} \neg\neg((\neg a \Rightarrow \neg b) \Rightarrow (b \Rightarrow a))$$

Solution We did work it out in class and in the book.

QUESTION 8 Use the heuristic method defined in chapter 11 to prove that

$$\not\vdash_{\mathbf{LI}}((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)).$$

Solution: To prove that our formula is not provable in **LI** we construct its possible decomposition trees following our heuristic, discuss their relationship and show that each of them must have an non-axiom leaf.

First tree is as follows.

T1

$$\longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$

$$\mid (\rightarrow\Rightarrow)$$

$$\neg b, (\neg a \Rightarrow b) \longrightarrow a$$

$$\mid (exch \rightarrow)$$

$$(\neg a \Rightarrow b), \neg b, \longrightarrow a$$

$$\bigwedge (\Rightarrow\longrightarrow)$$

$\neg b \longrightarrow \neg a$	$b, \neg b \longrightarrow a$
$\mid (\rightarrow \neg)$	$\mid (exch \rightarrow)$
$a, \neg b \longrightarrow$	$\neg b, b \longrightarrow a$
$\mid (exch \rightarrow)$	$\mid (\rightarrow weak)$
$\neg b, a \longrightarrow$	$\neg b, b \longrightarrow$
$\mid (\neg \rightarrow)$	$\mid (\neg \rightarrow)$
$a \longrightarrow b$	$b \longrightarrow b$
$non - axiom$	$axiom$

T2

$$\longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$

$$\bigwedge(\Rightarrow\longrightarrow)$$

$$\longrightarrow \neg a$$

$$\mid (\rightarrow \neg)$$

$$a \longrightarrow$$

$$non - axiom$$

$$b \longrightarrow (\neg b \Rightarrow a)$$

$$\mid (\rightarrow\Rightarrow)$$

$$b, \neg b \longrightarrow a$$

$$\mid (exch \rightarrow)$$

$$\neg b, b \longrightarrow a$$

$$\mid (\rightarrow weak)$$

$$\neg b, b \longrightarrow$$

$$\mid (\neg \rightarrow)$$

$$b \longrightarrow b$$

$$axiom$$

Observe that **T1** and **T2** have identical subtrees ending with identical leaves.

Third tree is obtained by the third choice of the decomposition rule at the second node of the tree **T1**, namely the use of rule $(contr \rightarrow)$. This step produces a node

$$(\neg a \Rightarrow b), (\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$

Observe that next decomposition steps would give trees similar to **T1** and **T2**. We write down, as an example one of them, which follows the pattern of the tree **T1**.

T3

$$\longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$

$$\mid (contr \rightarrow)$$

$$(\neg a \Rightarrow b), (\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$

$$\mid (\rightarrow\Rightarrow)$$

$$\neg b, (\neg a \Rightarrow b), (\neg a \Rightarrow b) \longrightarrow a$$

$$\mid (exch \rightarrow)$$

$$(\neg a \Rightarrow b), \neg b, (\neg a \Rightarrow b) \longrightarrow a$$

$$\bigwedge (\Rightarrow\longrightarrow)$$

$$Left\ premiss$$

$$\neg b, (\neg a \Rightarrow b) \longrightarrow \neg a$$

$$\mid (\to \neg)$$

$$a, \neg b, (\neg a \Rightarrow b) \longrightarrow$$

$$\mid (exch \to)$$

$$\neg b, a, (\neg a \Rightarrow b) \longrightarrow$$

$$\mid (\neg \to)$$

$$a, (\neg a \Rightarrow b) \longrightarrow b$$

$$\mid (exch \to)$$

$$(\neg a \Rightarrow b), a \longrightarrow b$$

$$\bigwedge (\Rightarrow \longrightarrow)$$

$a \longrightarrow \neg a$ $\qquad\qquad$ $b, a \longrightarrow b$

$\mid (\to \neg)$ $\qquad\qquad$ $axiom$

$a, a \longrightarrow$

$non - axiom$

$$Right\ premiss$$

$$b, \neg b, (\neg a \Rightarrow b) \longrightarrow a$$

$$\mid (exch \to)$$

$$\neg b, b, (\neg a \Rightarrow b) \longrightarrow a$$

$$\mid (\to weak)$$

$$\neg b, b, (\neg a \Rightarrow b) \longrightarrow$$

$$\mid (\neg \to)$$

$$b, (\neg a \Rightarrow b) \longrightarrow b$$

$$axiom$$

Observe that the rule ($contr \rightarrow$) didn't and will never bring information to the tree construction which would replace a non-axiom leaf by an axiom leaf.

Next tree can be obtained by exploring second choice at the node 3 of the first tree.

T4

$$\longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$

$$\mid (\rightarrow\Rightarrow)$$

$$\neg b, (\neg a \Rightarrow b) \longrightarrow a$$

$$\mid (\rightarrow weak)$$

$$\neg b, (\neg a \Rightarrow b) \longrightarrow$$

$$\mid (\neg \rightarrow)$$

$$(\neg a \Rightarrow b) \longrightarrow b$$

$$\bigwedge (\Rightarrow\longrightarrow)$$

$\longrightarrow \neg a$	$b \longrightarrow b$
$\mid (\rightarrow \neg)$	$axiom$
$a \longrightarrow$	
$non - axiom$	

Observe that here again the rule $(contr \rightarrow)$ applied to any node to the tree $\mathbf{T4}$ would never gives us a possibility of replacing a non-axiom leaf by an axiom leaf.

Conclusion All possible decomposition trees will always contain a non- axiom leaf what ends the proof.

GENERAL REMARK We are using the word "PROOF" in two distinct senses.

In the first sense, we use it as a *formal proof* within a fixed proof system, namely **LI** and is represented as a proof tree, or sequence of expressions of the language $\mathcal{L}$ of **LI**.

In the second sense, it also designates certain sequences of sentences of English language (supplemented by some technical terms, if needed) that are supposed to serve as an argument justifying some assertions about the language $\mathcal{L}$, or proof system based on it.

In general, the language we are studying, in this case $\mathcal{L}$, is called **an OBJECT LANGUAGE**.

The language in which we formulate and prove the results about the object language is called **the METALANGUAGE**. The metalanguage might also be formalized and made the object of study, which we would carry in **a meta-metalanguage**.

We use English as our not formalized metalanguage, although, we use only a mathematically weak portion of the English language. enddescription

The contrast between the language and metalanguage is also present in study for example, a foreign language. In French study class, French is the object language, while the metalanguage, the language we use, is English.

The distinction between **proof** and **meta-proof**, i.e. a proof in the metalanguage, is as follows.

We construct (in the metalanguage) a decomposition tree which is **a formal proof in the object language.**

By doing so, we prove in the metalanguage, that the proof in the object language exists.

Such proof is called **a meta-proof**, and the fact thus proved is called a **meta-theorem**.