CHAPTER 12

Gentzen Proof System for Intuitionistic Logic

In 1935 G.Gentzen formulated a first syntactically decidable formalization for classical and intuitionistic logic and proved its equivalence with the Heyting's original Hilbert style formalization (the famous Gentzen's Hauptsatz). We present here the original version of his work and discuss his original proof of the Hauptsatz Theorem. We deal here, as it has happened historically, with proof theoretical formalizations of the intuitionistic logic only. It means we present here the intuitionistic logic as a proof system only, as it was done in the original papers and leave the model theoretic investigations for later.

1 LI - The Gentzen Sequent Calculus

The proof system **LI** was published by Gentzen in 1935 as a particular case of his proof system **LK** for the classical logic. We have already discussed a version of the original Gentzen's system **LK** in the previous chapter **??**, so we present here the proof system **LI** first and then we show how it can be extended to the original Gentzen system **LK**.

Language of LI

Let $SEQ = \{\ \Gamma \longrightarrow \Delta :\ \Gamma, \Delta \in \mathcal{F}^*\ \}$ be the set of all Gentzen sequents built out of the formulas of the language

$$\mathcal{L} = \mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg\}}$$

and the additional symbol $\longrightarrow$.

In the intuitionistic logic we deal only with sequents of the form $\Gamma \longrightarrow \Delta$, where Δ consists of at most one formula. I.e. we assume that all sequents are elements of a following subset IS of the set SEQ of all sequents.

$$IS = \{\Gamma \longrightarrow \Delta : \quad \Delta \ \ consists\ of\ at\ most\ one\ formula\}. \tag{1}$$

The set IS is called the set of all intuitionistic sequents.

Axioms of LI

As the axioms of **LI** we adopt any sequent from the set IS defined by (1), which contains a formula that appears on both sides of the sequent arrow $\longrightarrow$, i.e any sequent of the form

$$\Gamma_1, A, \Gamma_2 \ \longrightarrow \ A, \tag{2}$$

for any formula $A \in \mathcal{F}$ and any sequences $\Gamma_1, \Gamma_2 \in \mathcal{F}^*$.

Inference rules of LI

The set inference rules is divided into two groups: the structural rules and the logical rules. They are defined as follows.

Structural Rules of LI

Weakening

$$(\rightarrow weak) \ \frac{\Gamma \ \longrightarrow}{\Gamma \ \longrightarrow \ A} \ .$$

A is called the weakening formula.

Contraction

$$(contr \rightarrow) \ \frac{A, A, \Gamma \ \longrightarrow \ \Delta}{A, \Gamma \ \longrightarrow \ \Delta},$$

A is called the contraction formula , Δ contains at most one formula.

Exchange

$$(exchange \rightarrow) \ \frac{\Gamma_1, A, B, \Gamma_2 \ \longrightarrow \ \Delta}{\Gamma_1, B, A, \Gamma_2 \ \longrightarrow \ \Delta},$$

Δ contains at most one formula.

Logical Rules of LI

Conjunction rules

$$(\cap \rightarrow) \ \frac{A, B, \Gamma \ \longrightarrow \ \Delta}{(A \cap B), \Gamma \ \longrightarrow \ \Delta}, \qquad (\rightarrow \cap) \ \frac{\Gamma \ \longrightarrow \ A \ ; \ \Gamma \ \longrightarrow \ B}{\Gamma \ \longrightarrow \ (A \cap B)},$$

Δ contains at most one formula.

Disjunction rules

$$(\rightarrow \cup)_1 \ \frac{\Gamma \ \longrightarrow \ A}{\Gamma \ \longrightarrow \ (A \cup B)}, \qquad (\rightarrow \cup)_2 \ \frac{\Gamma \ \longrightarrow \ B}{\Gamma \ \longrightarrow \ (A \cup B)},$$

$$(\cup \rightarrow) \quad \frac{A, \Gamma \longrightarrow \Delta \; ; \; B, \Gamma \longrightarrow \Delta}{(A \cup B), \Gamma \longrightarrow \Delta},$$

Δ contains at most one formula.

Implication rules

$$(\rightarrow\Rightarrow) \quad \frac{A, \Gamma \longrightarrow B}{\Gamma \longrightarrow (A \Rightarrow B)}, \qquad (\Rightarrow\rightarrow) \quad \frac{\Gamma \longrightarrow A \; ; \; B, \Gamma \longrightarrow \Delta}{(A \Rightarrow B), \Gamma \longrightarrow \Delta},$$

Δ contains at most one formula.

Negation rules

$$(\neg \rightarrow) \quad \frac{\Gamma \longrightarrow A}{\neg A, \Gamma \longrightarrow}, \qquad (\rightarrow \neg) \quad \frac{A, \Gamma \longrightarrow}{\Gamma \longrightarrow \neg A}.$$

Formally we define:

$$\mathbf{LI} = (\mathcal{L},\ IS,\ AX,\ \mathit{Structural\ rules},\ \ \mathit{Logical\ rules}\}),$$

where IS is defined by (1), Structural rules and Logical rules are the inference rules defined above, and AX is the axiom of the system defined by the schema (2).

We write

$$\vdash_{LI} \Gamma \longrightarrow \Delta$$

to denote that the sequent $\Gamma \longrightarrow \Delta$ has a proof in **LI**.

We say that a formula $A \in \mathcal{F}$ has a proof in **LI** and write it as

$$\vdash_{LI} A$$

when the sequent $\longrightarrow A$ has a proof in **LI**, i.e.

$$\vdash_{LI} A \;\; \mathit{if\ and\ only\ if} \;\; \vdash_{LI} \; \longrightarrow A.$$

2 Decomposition Trees in LI

Search for proofs in **LI** is a much more complicated process then the one in classical logic systems **RS** or **GL**.

Here, as in any other Gentzen style proof system, proof search procedure consists of building the decomposition trees.

In **RS** the decomposition tree $\mathbf{T}_A$ of any formula A, and hence of any sequence Γ is always unique.

In **GL** the "blind search" defines, for any formula A a finite number of decomposition trees, but it can be proved that the search can be reduced to examining only one of them, due to the absence of structural rules.

In **LI** the structural rules play a vital role in the proof construction and hence, in the proof search. We consider here a number of examples to show the complexity of the problem of examining possible decomposition trees for a given formula A. We are going to see that the fact that a given decomposition tree ends with an axiom leaf does not always imply that the proof does not exist. It might only imply that our search strategy was not good. Hence the problem of deciding whether a given formula A does, or does not have a proof in **LI** becomes more complex then in the case of Gentzen system for classical logic.

Before we define a heuristic method of searching for proof and deciding whether such a proof exists or not in **LI** we make some observations.

Observation 1: the logical rules of **LI** are similar to those in Gentzen type classical formalizations we examined in previous chapters in a sense that each of them introduces a logical connective.

Observation 2: The process of searching for a proof is, as before a decomposition process in which we use the inverse of logical and structural rules as decomposition rules.

For example the implication rule:

$$(\rightarrow\Rightarrow)\ \ \frac{A, \Gamma \ \longrightarrow\ B}{\Gamma \ \longrightarrow\ (A \Rightarrow B)}$$

becomes an **implication decomposition rule** (we use the same name $(\rightarrow\Rightarrow)$ in both cases)

$$(\rightarrow\Rightarrow)\ \ \frac{\Gamma \ \longrightarrow\ (A \Rightarrow B)}{A, \Gamma \ \longrightarrow\ B}.$$

Observation 3: we write our proofs in as trees, instead of sequences of expressions, so the proof search process is a process of building a decomposition tree. To facilitate the process we write, as before, the decomposition rules, structural rules included in a "tree " form.

For example the the above implication decomposition rule is written as follows.

$$\Gamma \;\longrightarrow\; (A \Rightarrow B)$$

$$\mid (\rightarrow\Rightarrow)$$

$$A, \Gamma \;\longrightarrow\; B$$

The two premisses implication rule $(\Rightarrow\rightarrow)$ written as the tree decomposition rule becomes

$$(A \Rightarrow B), \Gamma \;\longrightarrow$$

$$\bigwedge (\Rightarrow\rightarrow)$$

$$\Gamma \;\longrightarrow\; A \qquad\qquad B, \Gamma \;\longrightarrow$$

For example the structural weakening rule is written as the decomposition rule is written as

$$(\rightarrow weak)\ \frac{\Gamma \;\longrightarrow\; A}{\Gamma \;\longrightarrow}$$

We write it in a tree form as follows.

$$\Gamma \;\longrightarrow\; A$$

$$\mid (\rightarrow weak)$$

$$\Gamma \;\longrightarrow$$

We define, as before the notion of decomposable and indecomposable formulas and sequents as follows.

Decomposable formula is any formula of the degree ≥ 1.

Decomposable sequent is any sequent that contains a decomposable formula.

Indecomposable formula is any formula of the degree 0, i.e. any propositional variable.

Indecomposable sequent is a sequent formed from indecomposable formulas only.

Decomposition tree construction (1): given a formula $A \in$ we construct its decomposition tree $\mathbf{T_A}$ as follows.

Root of the tree is the sequent $\longrightarrow A$.

Given a node n of the tree we identify a decomposition rule applicable at this node and write its premisses as the leaves of the node n.

We stop the decomposition process when we obtain an axiom or all leaves of the tree are indecomposable.

Observation 4: the decomposition tree $\mathbf{T_A}$ obtained by the construction (1) most often is not unique.

Observation 5: the fact that we find a decomposition tree $\mathbf{T_A}$ with non-axiom leaf does not mean that $\nvdash_{LI} A$. This is due to the role of structural rules in **LI** and will be discussed later in the chapter.

We illustrate the problems arising with proof search procedures, i.e. decomposition trees construction in the next section 3 and give a heuristic proof searching procedure in the section 4.

3 Proof Search Examples

We perform proof search and decide the existence of proofs in **LI** for a given formula $A \in \mathcal{F}$ by constructing its decomposition trees $\mathbf{T_A}$. We examine here some examples to show the complexity of the problem.

Example 1

Determine whether $\quad \vdash_{\mathbf{LI}} \longrightarrow A$ for $A = ((\neg A \cap \neg B) \Rightarrow \neg(A \cup B))$.

This means that we have to construct some, or all decomposition trees of

$$\longrightarrow ((\neg A \cap \neg B) \Rightarrow \neg(A \cup B)).$$

If we find a decomposition tree such that all its leaves are axiom, we have a proof.

If all possible decomposition trees have a non-axiom leaf, proof of A in **LI** does not exist.

Consider the following decomposition tree of $\longrightarrow A$.

$$\mathbf{T1_A}$$

$$\longrightarrow ((\neg A \cap \neg B) \Rightarrow (\neg(A \cup B))$$

$$| (\longrightarrow \Rightarrow)$$

$$(\neg A \cap \neg B) \longrightarrow \neg(A \cup B)$$

$$| (\longrightarrow \neg)$$

$$(A \cup B), (\neg A \cap \neg B) \longrightarrow$$

$$| (exch \longrightarrow)$$

$$(\neg A \cap \neg B), (A \cup B) \longrightarrow$$

$$| (\cap \longrightarrow)$$

$$\neg A, \neg B, (A \cup B) \longrightarrow$$

$$| (\neg \longrightarrow)$$

$$\neg B, (A \cup B) \longrightarrow A$$

$$| (\longrightarrow weak)$$

$$\neg B, (A \cup B) \longrightarrow$$

$$| (\neg \longrightarrow)$$

$$(A \cup B) \longrightarrow B$$

$$\bigwedge (\cup \longrightarrow)$$

$$A \longrightarrow B \qquad\qquad B \longrightarrow B$$

$$non - axiom \qquad\qquad axiom$$

The tree $\mathbf{T1_A}$ has a non-axiom leaf, so it does not constitute a proof in **LI**. But this fact does not yet prove that proof doesn't exist, as the decomposition tree in **LI** is not always unique.

Let's consider now the following tree.

$$\mathbf{T2_A}$$

$$\longrightarrow ((\neg A \cap \neg B) \Rightarrow (\neg(A \cup B))$$

$$| (\longrightarrow \Rightarrow)$$

$$(\neg A \cap \neg B) \longrightarrow \neg(A \cup B)$$

$$| (\longrightarrow \neg)$$

$$(A \cup B), (\neg A \cap \neg B) \longrightarrow$$

$$| (exch \longrightarrow)$$

$$(\neg A \cap \neg B), (A \cup B) \longrightarrow$$
$$\mid (\cap \longrightarrow)$$
$$\neg A, \neg B, (A \cup B) \longrightarrow$$
$$\mid (exch \longrightarrow)$$
$$\neg A, (A \cup B), \neg B \longrightarrow$$
$$\mid (exch \longrightarrow)$$
$$(A \cup B), \neg A, \neg B \longrightarrow$$
$$\bigwedge (\cup \longrightarrow)$$

$$A, \neg A, \neg B \longrightarrow \qquad\qquad B, \neg A, \neg B \longrightarrow$$
$$\mid (exch \longrightarrow) \qquad\qquad \mid (exch \longrightarrow)$$
$$\neg A, A, \neg B \longrightarrow \qquad\qquad B, \neg B, \neg A \longrightarrow$$
$$\mid (\neg \longrightarrow) \qquad\qquad \mid (exch \longrightarrow)$$
$$A, \neg B \longrightarrow A \qquad\qquad \neg B, B, \neg A \longrightarrow$$
$$axiom \qquad\qquad \mid (\neg \longrightarrow)$$
$$B, \neg A \longrightarrow B$$
$$axiom$$

All leaves of $\mathbf{T2_A}$ are axioms, what proves that $\mathbf{T2_A}$ is a proof of A and hence we proved that
$$\vdash_{\mathbf{LI}} ((\neg A \cap \neg B) \Rightarrow \neg(A \cup B)).$$

Example 2

Part 1: Prove that
$$\vdash_{\mathbf{LI}} \ \longrightarrow (A \Rightarrow \neg\neg A),$$

Part 2: Prove that
$$\nvdash_{\mathbf{LI}} \ \longrightarrow (\neg\neg A \Rightarrow A).$$

Solution of Part 1
To prove that
$$\vdash_{\mathbf{LI}} \ \longrightarrow (A \Rightarrow \neg\neg A)$$
we have to construct some, or all decomposition trees of
$$\longrightarrow (A \Rightarrow \neg\neg A).$$

The tree that ends with all axioms leaves is a proof of $(A \Rightarrow \neg\neg A)$ in **LI**.

Consider the following decomposition tree of $\longrightarrow A$, for $A = (\neg\neg A \Rightarrow A)$..

$$\mathbf{T}_A$$

$$\longrightarrow (A \Rightarrow \neg\neg A).$$
$$\mid (\longrightarrow\Rightarrow)$$
$$A \longrightarrow \neg\neg A$$
$$\mid (\longrightarrow \neg)$$
$$\neg A, A \longrightarrow$$
$$\mid (\neg \longrightarrow)$$
$$A \longrightarrow A$$
$$axiom$$

All leaves of $\mathbf{T}_A$ are axioms what proves that $\mathbf{T}_A$ is a proof of $\longrightarrow (A \Rightarrow \neg\neg A)$ and we don't need to construct other decomposition trees.

Solution of Part 2
To prove that

$$\nvdash_{\mathbf{LI}} \; \longrightarrow (\neg\neg A \Rightarrow A)$$

we have to construct all decomposition trees of $(A \Rightarrow \neg\neg A)$ and show that each of them has an non-axiom leaf.

Consider the first decomposition tree defined as follows.

$$\mathbf{T1}_A$$

$$\longrightarrow (\neg\neg A \Rightarrow A)$$
$$first\ of\ 2\ choices : (\rightarrow\Rightarrow), (\rightarrow weak)$$
$$\mid (\rightarrow\Rightarrow)$$
$$\neg\neg A \longrightarrow A$$
$$first\ of\ 2\ choices : (\rightarrow weak), (contr \rightarrow)$$

$$| (\rightarrow weak)$$

$$\neg\neg A \longrightarrow$$

$$first\ of\ 2\ choices : (\neg \rightarrow), (contr \rightarrow)$$

$$| (\neg \rightarrow)$$

$$\longrightarrow \neg A$$

$$first\ of\ 2\ choices : (\neg \rightarrow), (\rightarrow weak)$$

$$| (\rightarrow \neg)$$

$$A \longrightarrow$$

$$indecomposable$$

$$non - axiom$$

We use the first tree created to define all other possible decomposition trees by exploring the alternative search paths as indicated at the nodes of the tree.

T1$_A$

$$\longrightarrow (\neg\neg A \Rightarrow A)$$

$$| (\longrightarrow\Rightarrow)$$

$$one\ of\ 2\ choices$$

$$\neg\neg A \longrightarrow A$$

$$]$$

$$| (contr \longrightarrow)$$

$$second\ of\ 2\ choices$$

$$\neg\neg A, \neg\neg A \longrightarrow A$$

$$| (\longrightarrow weak)$$

$$first\ of\ 2\ choices$$

$$\neg\neg A, \neg\neg A \longrightarrow$$

$$| (\neg \longrightarrow)$$

$$first\ of\ 2\ choices$$

$$\neg\neg A \longrightarrow \neg A$$

$$| (\longrightarrow \neg)$$

$$the\ only\ choice$$

$$A, \neg\neg A \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$the\ only\ choice$$

$$\neg\neg A, A \longrightarrow$$

$$\mid (\longrightarrow \neg)$$

$$the\ only\ choice$$

$$A \longrightarrow \neg A$$

$$\mid (\longrightarrow \neg)$$

$$first\ of\ 2\ choices$$

$$A, A \longrightarrow$$

$$indecomposable$$

$$non - axiom$$

We can see from the above decomposition trees that the "blind" construction of all possible trees only leads to more complicated trees, due to the presence of structural rules. Observe that the "blind" application of ($contr \longrightarrow$) gives an infinite number of decomposition trees. To decide that none of them will produce a proof we need some extra knowledge about patterns of their construction, or just simply about the number useful of application of structural rules within the proofs.

In this case we can just make an "external" observation that the our first tree $\mathbf{T1}_A$ is in a sense a minimal one; that all other trees would only complicate this one in an inessential way, i.e. we will never produce a tree with all axioms leaves.

One can formulate a deterministic procedure giving a finite number of trees, but the proof of its correctness require some extra knowledge. We are going to discuss a motivation and an heuristics for the proof search in the next section.

Within the scope of this book we accept the "external" explanation for the heuristics we use as a sufficient solution.

As we can see from the above examples structural rules and especially the ($contr \rightarrow$) rule complicates the proof searching task. Both Gentzen type proof systems **RS** and **GL** from the previous chapter don't contain the structural rules and are complete with respect to classical semantics, as is the original Gentzen system **LK**, which does contain the structural rules. As (via Completeness Theorem) all three classical proof system **RS, GL, LK** are equivalent we can say that the structural rules can be eliminated from the system **LK**.

A natural question of elimination of structural rules from the intutionistic Gentzen system **LI** arizes.

The following example illustrates the negative answer.

Example 3

We know, by the theorem about the connection between classical and intuitionistic logic (theorem **??**) and corresponding Completeness Theorems that for any formula $A \in \mathcal{F}$,

$$\models A \ \ if\ and\ only\ if\ \ \vdash_I \neg\neg A,$$

where $\models A$ means that A is a classical tautology, $\vdash_I$ means that A is intutionistically provable, i.e. is provable in any intuitionistically complete proof system. The system **LI** is intuitionistically complete, so we have that for any formula A,

$$\models A \ \ if\ and\ only\ if\ \ \vdash_{\mathbf{LI}} \neg\neg A.$$

We have just proved that $\nvdash_{\mathbf{LI}} (\neg\neg A \Rightarrow A)$. Obviously $\models (\neg\neg A \Rightarrow A)$, so we know that $\neg\neg(\neg\neg A \Rightarrow A)$ must have a proof in **LI**.

We are going to prove that

$$\vdash_{\mathbf{LI}} \neg\neg(\neg\neg A \Rightarrow A)$$

and that the structural rule $(contr \longrightarrow)$ is essential to the existence of its proof, i.e. that without it the formula $\neg\neg(\neg\neg A \Rightarrow A)$ is not provable in **LI**.

The following decomposition tree $\mathbf{T}_A$ is a proof of $A = \neg\neg(\neg\neg A \Rightarrow A)$ in **LI**.

$$\mathbf{T}_A$$

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$

$$first\ of\ 2\ choices:\ (\rightarrow \neg), (\rightarrow weak)$$

$$\mid (\longrightarrow \neg)$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$first\ of\ 2\ choices: (contr \longrightarrow), (\neg \longrightarrow)$$

$$\mid (contr \longrightarrow)$$

$$\neg(\neg\neg A \Rightarrow A), \neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$one\ of\ 2\ choices$$

$$\mid (\neg \longrightarrow)$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow (\neg\neg A \Rightarrow A)$$

$$one\ of\ 3\ choices$$

$$\mid (\longrightarrow\Rightarrow)$$

$$\neg(\neg\neg A \Rightarrow A), \neg\neg A \longrightarrow A$$

$$one\ of\ 2\ choices$$

$$\mid (\longrightarrow weak)$$

$$\neg(\neg\neg A \Rightarrow A), \neg\neg A \longrightarrow$$

$$one\ of\ 3\ choices$$

$$\mid (exch \longrightarrow)$$

$$\neg\neg A, \neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$one\ of\ 3\ choices$$

$$\mid (\neg \longrightarrow)$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow \neg A$$

$$one\ of\ 3\ choices$$

$$\mid (\longrightarrow \neg)$$

$$A, \neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$one\ of\ 2\ choices$$

$$\mid (exch \longrightarrow)$$

$$\neg(\neg\neg A \Rightarrow A), A \longrightarrow$$

$$one\ of\ 3\ choices$$

$$\mid (\neg \longrightarrow)$$

$$A \longrightarrow (\neg\neg A \Rightarrow A)$$

$$one\ of\ 3\ choices$$

$$\mid (\longrightarrow\Rightarrow)$$

$$\neg\neg A, A \longrightarrow A$$

$$axiom$$

Assume now that the rule ($contr \longrightarrow$) is not available. The decomposition possible decomposition trees are as follows.

$$\mathbf{T1}_A$$

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \neg)$$

$$one\ of\ 2\ choices$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$only\ one\ choice$$

$$\longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow\Rightarrow)$$

$$one\ of\ 2\ choices$$

$$\neg\neg A \longrightarrow A$$

$$\mid (\longrightarrow weak)$$

$$only\ one\ choice$$

$$\neg\neg A \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$only\ one\ choice$$

$$\longrightarrow \neg A$$

$$\mid (\longrightarrow \neg)$$

$$one\ of\ 2\ choices$$

$$A \longrightarrow$$

$$non - axiom$$

T2$_A$

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow weak)$$

$$second\ of\ 2\ choices$$

$$\longrightarrow$$

$$non - axiom$$

T3$_A$

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$
$$\mid (\longrightarrow \neg)$$
$$\neg(\neg\neg A \Rightarrow A) \longrightarrow$$
$$\mid (\neg \longrightarrow)$$
$$\longrightarrow (\neg\neg A \Rightarrow A)$$
$$\mid (\longrightarrow weak)$$
$$second\ of\ 2\ choices$$
$$\longrightarrow$$
$$non-axiom$$

T4$_A$

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$
$$\mid (\longrightarrow \neg)$$
$$\neg(\neg\neg A \Rightarrow A) \longrightarrow$$
$$\mid (\neg \longrightarrow)$$
$$\longrightarrow (\neg\neg A \Rightarrow A)$$
$$\mid (\longrightarrow\Rightarrow)$$
$$]$$
$$\neg\neg A \longrightarrow A$$
$$\mid (\longrightarrow weak)$$
$$only\ one\ choice$$
$$\neg\neg A \longrightarrow$$
$$\mid (\neg \longrightarrow)$$
$$only\ one\ choice$$
$$\longrightarrow \neg A$$
$$\mid (\longrightarrow weak)$$
$$second\ of\ 2\ choices$$
$$\longrightarrow$$
$$non-axiom$$

This proves that the formula $\neg\neg(\neg\neg A \Rightarrow A)$ is not provable in **LI** without ($contr \longrightarrow$) rule, i.e. that this rule can't be eliminated.

4 Proof Search Heuristic Method

Before we define a heuristic method of searching for proof in **LI** let's make some additional observations to the observations 1-5 from section 2.

Observation 6: Our goal while constructing the decomposition tree is to obtain axiom or indecomposable leaves. With respect to this goal the use logical decomposition rules has a priority over the use of the structural rules and we use this information while describing the proof search heuristic.

Observation 7: all logical decomposition rules ($\circ \rightarrow$), where $\circ$ denotes any connective, must have a formula we want to decompose as the first formula at the decomposition node, i.e. if we want to decompose a formula $\circ A$, the node must have a form $\circ A, \Gamma \longrightarrow \Delta$. Sometimes it is necessary to decompose a formula within the sequence Γ first in order to find a proof.

For example, consider two nodes

$$n_1 = \neg\neg A, (A \cap B) \longrightarrow B$$

and

$$n_2 = (A \cap B), \neg\neg A \longrightarrow B.$$

We are going to see that the results of decomposing n_1 and n_2 differ dramatically.

Let's decompose the node n_1. Observe that the only way to be able to decompose the formula $\neg\neg A$ is to use the rule ($\rightarrow weak$) first. The two possible decomposition trees that starts at the node n_1 are as follows.

$$\mathbf{T1}_{n_1}$$

$$\neg\neg A, (A \cap B) \longrightarrow B$$
$$\mid (\rightarrow weak)$$
$$\neg\neg A, (A \cap B) \longrightarrow$$
$$\mid (\neg \rightarrow)$$
$$(A \cap B) \longrightarrow \neg A$$
$$\mid (\cap \rightarrow)$$
$$A, B \longrightarrow \neg A$$
$$\mid (\rightarrow \neg)$$
$$A, A, B \longrightarrow$$
$$non - axiom$$

$\mathbf{T2}_{n_1}$

$$\neg\neg A, (A \cap B) \longrightarrow B$$

$$\mid (\rightarrow weak)$$

$$\neg\neg A, (A \cap B) \longrightarrow$$

$$\mid (\neg \rightarrow)$$

$$(A \cap B) \longrightarrow \neg A$$

$$\mid (\rightarrow \neg)$$

$$A, (A \cap B) \longrightarrow$$

$$\mid (\cap \rightarrow)$$

$$A, A, B \longrightarrow$$

$$non - axiom$$

Let's now decompose the node n_2. Observe that following our **Observation 6** we start by decomposing the formula $(A \cap B)$ by the use of the rule $(\cap \rightarrow)$ first. A decomposition tree that starts at the node n_2 is as follows.

$\mathbf{T}_{n_2}$

$$(A \cap B), \neg\neg A \longrightarrow B$$

$$\mid (\cap \rightarrow)$$

$$A, B, \neg\neg A \longrightarrow B$$

$$axiom$$

This proves that the node n_2 is provable in **LI**, i.e.

$$\vdash_{\mathbf{LI}} \quad (A \cap B), \neg\neg A \longrightarrow B.$$

Of course, we have also that the node n_1 is also provable in **LI**, as one can obtain the node n_2 from it by the use of the rule $(exch \rightarrow)$.

Observation 8: the use of structural rules are important and necessary while we search for proofs. Nevertheless we have to use them on the "must" basis and set up some guidelines and priorities for their use.

For example, use of weakening rule discharges the weakening formula, and hence an information that may be essential to the proof. We should use it only when it is absolutely necessary for the next decomposition steps. Hence, the use of weakening rule ($\rightarrow weak$) can, and should be restricted to the cases when it leads to possibility of the use of the negation rule ($\neg \rightarrow$). This was the case of the decomposition tree $\mathbf{T1}_{n_1}$. We used it as an necessary step, but still it discharged too much information and we didn't get a proof, when proof of the node existed.

In this case the first rule in our search should have been the exchange rule, followed by the conjunction rule (no information discharge) not the weakening (discharge of information) followed by negation rule. The full proof of the node n_1 is the following.

$\mathbf{T3}_{n_1}$

$$\neg\neg A, (A \cap B) \longrightarrow B$$

$$\mid (exch \longrightarrow)$$

$$(A \cap B), \neg\neg A \longrightarrow B$$

$$\mid (\cap \rightarrow)$$

$$A, B, \neg\neg A \longrightarrow B$$

$$axiom$$

As a result of the **observations 1- 5** from section 2 and **observations 6 - 8** above we adopt the following.

Heuristic Procedure for Proof Search in LI.

For any $A \in \mathcal{F}$ we construct the decomposition trees $\mathbf{T}_{\rightarrow A}$ following the rules below.

Decomposition Tree Generation rules.

1. Use first logical rules where applicable.

2. Use $(exch \rightarrow)$ rule to decompose, via logical rules, as many formulas on the left side of $\longrightarrow$ as possible.

3. Use $(\rightarrow weak)$ only on a "must" basis in connection with $(\neg \rightarrow)$ rule.

4. Use $(contr \rightarrow)$ rule as the last recourse and only to formulas that contain $\neg$ or $\Rightarrow$ as connectives.

5. Let's call a formula A to which we apply $(contr \rightarrow)$ rule **a contraction** formula.

6. The only contraction formulas are formulas containing $\neg$ or $\Rightarrow$ between theirs logical connectives.

7. Within the process of construction of all possible trees use $(contr \rightarrow)$ rule only to contraction formulas.

8. Let C be a contraction formula appearing on the node n of the decomposition tree of $\mathbf{T}_{\rightarrow A}$. For any contraction formula C, any node n, we apply $(contr \rightarrow)$ rule the the formula C at most as many times as the number of sub-formulas of C.

If we find a tree with all axiom leaves we have a proof, i.e. $\vdash_{LI} A$ and if all (finite number) trees have a non-axiom leaf we have proved that proof of A does not exist, i.e. $\nvdash_{LI} A$.

5 LK - Original Gentzen system for the classical logic

In the original Gentzen work the intuitionistic logic is obtained as a particular case of his system **LK** for classical logic. We deal here with a propositional part of the original formalization.

The differences are as follows.

Language of LK

In classical case we adopt the whole set

$$SEQ = \{\Gamma \longrightarrow \Delta : \quad \Gamma, \Delta \in \mathcal{F}^* \tag{3}$$

instead of its subset IS defined by (1), i.e. as follows.

$$IS = \{\Gamma \longrightarrow \Delta : \quad \Delta \ consists\ of\ at\ most\ one\ formula\}$$

adopted for the intuitionistic logic.

Axioms of LK

We adopt the set of axioms, relevant to the difference of the language. I.e. in the classical case we have as an axiom any sequent of the form

$$\Gamma_1, A, \Gamma_2 \longrightarrow \Gamma_3, A, \Gamma_4 \tag{4}$$

while it becomes

$$\Gamma_1, A, \Gamma_2 \longrightarrow A$$

in the case of the intuitionistic logic.

Rules of inference of LK

The changes are as follows.

1. There two more structural rules in the system **LK**. One more contraction rule:

$$(\rightarrow contr) \quad \frac{\Gamma \;\longrightarrow\; \Delta, A, A,}{\Gamma \;\longrightarrow\; \Delta, A},$$

and one more exchange rule:

$$(\rightarrow exchange) \quad \frac{\Delta \longrightarrow \Gamma_1, A, B, \Gamma_2}{\Delta \longrightarrow \Gamma_1, B, A, \Gamma_2}.$$

Observe that they both become obsolete in the language of the intuitionistic logic.

2. In all inference rules we drop the intuitionistic restriction that the sequence Δ in the succedant of the sequence is empty.

3. In particular, we add the sequence Δ to all succedants which were empty or one formula in all of the rules of **LI**.

The rules of inference of **LK** are hence as follows.

Structural Rules of LK

Weakening

$$(weakening \rightarrow) \quad \frac{\Gamma \;\longrightarrow\; \Delta}{A, \Gamma \;\longrightarrow\; \Delta}, \qquad (\rightarrow weakening) \quad \frac{\Gamma \;\longrightarrow\; \Delta}{\Gamma \;\longrightarrow\; \Delta, A}\ .$$

A is called the weakening formula.

Contraction

$$(contr \rightarrow)\ \frac{A, A, \Gamma \longrightarrow \Delta}{A, \Gamma \longrightarrow \Delta}, \qquad (\rightarrow contr)\ \frac{\Gamma \longrightarrow \Delta, A, A,}{\Gamma \longrightarrow \Delta, A},$$

A is called the contraction formula.

Exchange

$$(exchange \rightarrow)\ \frac{\Gamma_1, A, B, \Gamma_2 \longrightarrow \Delta}{\Gamma_1, B, A, \Gamma_2 \longrightarrow \Delta}, \qquad (\rightarrow exchange)\ \frac{\Delta \longrightarrow \Gamma_1, A, B, \Gamma_2}{\Delta \longrightarrow \Gamma_1, B, A, \Gamma_2}.$$

Logical Rules of LK

Conjunction rules

$$(\cap \rightarrow)\ \frac{A, B, \Gamma \longrightarrow \Delta}{(A \cap B), \Gamma \longrightarrow \Delta}, \qquad (\rightarrow \cap)\ \frac{\Gamma \longrightarrow \Delta,\ A\ ;\ \Gamma \longrightarrow \Delta,\ B\Delta}{\Gamma \longrightarrow \Delta, (A \cap B)}.$$

Disjunction rules

$$(\rightarrow \cup)\ \frac{\Gamma \longrightarrow \Delta, A, B}{\Gamma \longrightarrow \Delta, (A \cup B)}, \qquad (\cup \rightarrow)\ \frac{A, \Gamma \longrightarrow \Delta\ ;\ B, \Gamma \longrightarrow \Delta}{(A \cup B), \Gamma \longrightarrow \Delta}.$$

Implication rules

$$(\rightarrow\Rightarrow)\ \frac{A, \Gamma \longrightarrow \Delta, B}{\Gamma \longrightarrow \Delta, (A \Rightarrow B)}, \qquad (\Rightarrow\rightarrow)\ \frac{\Gamma \longrightarrow \Delta, A\ ;\ B, \Gamma \longrightarrow \Delta}{(A \Rightarrow B), \Gamma \longrightarrow \Delta}.$$

Negation rules

$$(\neg \rightarrow)\ \frac{\Gamma \longrightarrow \Delta,\ A}{\neg A, \Gamma \longrightarrow \Delta}, \qquad (\rightarrow \neg)\ \frac{A, \Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, \neg A}.$$

Formally we define:

$$\mathbf{LK} = (SEQ,\ AX,\ \textbf{Structural rules of LK},\ \ \textbf{Logical rules of LK}\}),$$

where SEQ is defined by (1), **Structural rules of LK** , **Logical rules of LK** are the inference rules defined above, and AX is the axiom of the system defined by the schema (4).

5.1 Exercises and Homework

1. Give a formal proof **LI** axioms **A1 - A11** and all examples of intuitionistic tautologies from Chapter 10.

2. Show that none of the formulas from chapter 10 that are classical and not intuitionistic tautologies is provable in **LI**.

3. Find the formal proofs in **LI** of double negation of all of formulas from chapter 10 that are classical and not intuitionistic tautologies.

4. Give the proof of the Glivenko theorem, i.e. prove that any formula $A \in \mathcal{F}$, A is a classically provable if and only if $\neg\neg A$ is an intuitionistically provable.

5. Give examples of formulas illustrating that the following theorems hold.

Theorem 5.1 *(Gödel) For any $A, B \in \mathcal{F}$, a formula $(A \Rightarrow \neg B)$ is a classically provable if and only if it is an intuitionistically provable.*

Theorem 5.2 *(Gödel) If a formula A contains no connectives except $\cap$ and $\neg$, then A is a classically provable if and only if it is an intuitionistically provable.*

6. Give **LK** proofs of all formulas from chapter 10 that are not provable in **LI**.