CHAPTER 13

Gentzen Style Proof System for Classical Predicate Logic - The System QRS

Part Two

1 Completeness Theorem for QRS

Given a first order language $\mathcal{L}$ with the set of variables VAR and the set of formulas $\mathcal{F}$. We have defined a notion of a model and counter- model of a formula A of $\mathcal{L}$ as follows.

Definition 1.1 (Model) *A structure $\mathcal{M} = [M, I]$ is called a model of $A \in \mathcal{F}$ if and only if*

$$(\mathcal{M}, v) \models A$$

for all valuations $v : VAR \longrightarrow M$.

M is called a universe of the model, I the interpretation.

Definition 1.2 (Counter - Model) *A structure $\mathcal{M} = [M, I]$ is called a counter-model of $A \in \mathcal{F}$ if and only if there is a valuation $v : VAR \longrightarrow M$, such that*

$$(\mathcal{M}, v) \not\models A.$$

The definition of the first order logic tautology is the following.

Definition 1.3 (Tautology) *For any $A \in \mathcal{F}$, A is called a tautology and denoted by $\models A$, if and only if all structures $\mathcal{M} = [M, I]$ ate models of A, i.e.*

$$\models A \quad if\ and\ only\ if \quad (\mathcal{M}, v) \models A$$

for all structures $\mathcal{M} = [M, I]$ and all valuations $v : VAR \longrightarrow M$.

Directly from the above definition we get the following, simple fact.

Fact 1.1 (Not Tautology) *For any $A \in \mathcal{F}$, A is not a tautology ($\not\models A$) if and only if there is a counter - model $\mathcal{M} = [M, I]$ of A, i.e. we can define M, I, and v such that $([M, I], v) \not\models A$.*

As our proof system is fixed, we will continue to use the notation $\vdash A$ ($\vdash \Gamma$) to denote that a formula A (a sequence Γ) has a proof in **QRS**.

Our goal now is to prove the Completeness Theorem for **QRS**. We do it, as in the propositional case, in two steps. First, we will prove the Soundness Lemmma:

Lemma 1.1 (Soundness Lemmma for QRS) *For any $\Gamma \in \mathcal{F}^*$,*

$$if \quad \vdash \Gamma \quad then \quad \models \Gamma,$$

and in particular, for any $A \in \mathcal{F}$,

$$if \quad \vdash A \quad then \quad \models A.$$

The proof is by step by step verification, similar to the propositional case and is left as an exercise. To complete the proof of the following

Theorem 1.1 (Completeness Theorem for QRS) *For any $\Gamma \in \mathcal{F}^*$,*

$$\vdash \Gamma \quad if\ and\ only\ if \quad \models \Gamma,$$

and in particular, for any $A \in \mathcal{F}$,

$$\vdash A \quad if\ and\ only\ if \quad \models A.$$

we have to prove the inverse implication to the Soundness Lemmma. We prove the formula case only and show that the case of sequences can be reduced to the formula case. I.e. we prove that the implication: $If \quad \models A \quad then \quad \vdash A$ is true. We do it, as in the propositional case, by proving the opposite implication to it, instead. I. e. we prove that the implication:

$$If \quad \not\vdash A \quad then \quad \not\models A$$

is true. This means that we prove that for any formula A, if we know that from the fact that A does not have a proof in **QRS** ($\not\vdash A$), we will be able to define its counter- model. The counter- model is defined, as in the propositional case, via the proof search (decomposition) tree. As we know, each formula A, generates its unique decomposition tree $\mathcal{T}_A$ and A has a proof only if this tree is finite and all its end sequences (leaves) are axioms. It means that if $\not\vdash A$

then we have two cases to consider: tree $\mathcal{T}_A$ contains a leaf which is not axiom or is infinite. We will show how in both cases to construct a counter- model for A, determined by the infinite branch or non-axiom leaf of the decomposition tree $\mathcal{T}_A$. Before describing a general method of constructing the counter-models determined by the decomposition tree let's look at some examples. **Example 1**

Let's consider a particular case of the formula

$$(\exists x A(x) \Rightarrow \forall x A(x)),$$

i.e. let A be a formula

$$(\exists x(P(x) \cap R(x,y)) \Rightarrow \forall x(P(x) \cap R(x,y)))$$

for P , R one and two argument predicate symbols, respectively. The decomposition tree $\mathcal{T}_A$ is the following:

$$(\exists x(P(x) \cap R(x,y)) \Rightarrow \forall x(P(x) \cap R(x,y)))$$

$$\mid (\Rightarrow)$$

$$\neg\exists x(P(x) \cap R(x,y)), \forall x(P(x) \cap R(x,y))$$

$$\mid (\neg\exists)$$

$$\forall x\neg(P(x) \cap R(x,y)), \forall x(P(x) \cap R(x,y))$$

$$\mid (\forall)$$

$$\neg(P(x_1) \cap R(x_1,y)), \forall x(P(x) \cap R(x,y))$$

where x_1 is a first free variable in **??** such that x_1 does not appear in $\forall x\neg(P(x) \cap R(x,y)), \forall x(P(x) \cap R(x,y))$

$$\mid (\neg\cap)$$

$$\neg P(x_1), \neg R(x_1,y), \forall x(P(x) \cap R(x,y))$$

$$\mid (\forall)$$

$$\neg P(x_1), \neg R(x_1,y), (P(x_2) \cap R(x_2,y))$$

where x_2 is a first free variable in the sequence **??** such that x_2 does not appear in $\neg P(x_1), \neg R(x_1,y), \forall x(P(x) \cap R(x,y))$, the sequence **??** is one-to- one, hence $x_1 \neq x_2$

$$\bigwedge (\cap)$$

$$\neg P(x_1), \neg R(x_1,y), P(x_2) \qquad\qquad \neg P(x_1), \neg R(x_1,y), R(x_2,y)$$

$x_1 \neq x_2$, Non-axiom $\qquad$ $x_1 \neq x_2$, Non-axiom

There are two non-axiom leaves, to define a counter- model for A we need to chose only one of them, for example, let's choose

$$\neg P(x_1), \neg R(x_1, y), P(x_2).$$

We define a counter - model for A, i.e. a structure $\mathcal{M} = [M, I]$ and a valuation v, such that $(\mathcal{M}, v) \not\models A$ as follows.

1. $M = T$, i.e. the universe is the set of all terms of our language.

2. We define the relations P_I and R_I in the set of all terms T as follows: for any term $t \in T$,

$P_I(t)$ HOLDS iff the negation $\neg P(t)$ of the formula $P(t)$ appears on the non-axiom leaf, and $P_I(t)$ DOES NOT HOLD otherwise.

R_I is defined similarly: for any terms $t, s \in T$,

$R_I(t, s)$ HOLDS iff the negation $\neg R(t, s)$ of the formula $R(t, s)$ appears on the non-axiom leaf, and $R_I(t, s)$ DOES NOT HOLD otherwise.

It is easy to see that in particular case of our non-axiom leaf: $P_I(x_1)$ holds, $R(x_1, y)$ holds for any variable y, and $P(x_2)$ does not hold.

3. We define the valuation $v : VAR \longrightarrow T$ as IDENTITY, i.e., we put $v(x) = x$ for any $x \in VAR$.

Obviously, for such defined structure $[M, I]$ and valuation v we have that $([M, I], v) \models P(x_1)$, $([M, I], v) \models R(x_1, y)$, and $([M, I], v) \not\models P(x_2)$ and hence we obtain that

$$([M, I], v) \not\models \neg P(x_1), \neg R(x_1, y), P(x_2).$$

This proves that such defined structure $[M, I]$ is a counter model for a non-axiom leaf, and hence, by the fact that if one premiss of a rule of inference is false, so is the conclusion, it is a counter-model for all sequences on the branch which ends with this leaf, and hence in particular, it is a counter - model for A.

The case of the infinite tree is similar, even if a little bit more complicated. Observe first that the rule $(\exists)$ is the the only rule of inference (decomposition) which can "produce" an infinite branch. We first show how to construct the counter-model in the case of the simplest application of this rule, i.e. in the case of the formula

$$\exists x A(x)$$

where A is an one argument relational symbol. All other cases are similar to this one. The infinite branch $\mathcal{B}$in this case consists elements of the whole decomposition tree:

$$\exists x A(x)$$

$$| \, (\exists)$$

$$A(t_1), \exists x A(x)$$

where t_1 is the first term in the sequence **??**, such that $A(t_1)$ does not appear on the tree above $A(t_1), \exists x A(x)$

$$| \, (\exists)$$

$$A(t_1), A(t_2), \exists x A(x)$$

where t_2 is the first term in the sequence **??**, such that $A(t_2)$ does not appear on the tree above $A(t_1), A(t_2), \exists x A(x)$, i.e. $t_2 \neq t_1$

$$| \, (\exists)$$

$$A(t_1), A(t_2), A(t_3), \exists x A(x)$$

where t_3 is the first term in the sequence **??**, such that $A(t_3)$ does not appear on the tree above $A(t_1), A(t_2), A(t_3), \exists x A(x)$, i.e. $t_3 \neq t_2 \neq t_1$

$$| \, (\exists)$$

$$A(t_1), A(t_2), A(t_3), A(t_4), \exists x A(x)$$

$$| \, (\exists)$$

.....

$$| \, (\exists)$$

.....

i.e.

$$\mathcal{B} = \{\exists x A(x), A(t_1), A(t_2), A(t_2), A(t_4), ...\}$$

where $t_1, t_2, ...$ is a one - to one sequence of all elements of the set of terms T.

This means that the infinite branch $\mathcal{B}$ contains with the formula $\exists x A(x)$ all its instances $A(t)$, for all terms $t \in T$.

We define the structure $[M, I]$ and valuation v in a similar way as in the previous example, i.e. we take as the universe M the set of all terms T, we define A_I as follows: $A_I(t)$ HOLDS if $\neg A(t) \in \mathcal{B}$ and $A_I(t)$ DOES NOT HOLDS if $A(t) \in \mathcal{B}$. We take, as before, the identity function, as the valuation, $v(x) = x$ for any $x \in VAR$ and define the interpretation I for functional symbols as follows. For any constant c, we put $c_I = c$, for any variable x, $x_I = v(x) = x$, and for any n-argument functional symbol f, we have still to define $f_I : T^n \longrightarrow T$. Observe that by definition, the function f_I has to assign a certain term to a sequence of terms $t_1, t_2, ..., t_n$ and the interpretation I says how we do it. Let's define:

$$f_I(t_1, t_2, ..., t_n) = f(t_1, t_2, ..., t_n).$$

It is easy to see that for any formula $A(t) \in \mathcal{B}$,

$$([T, I], v) \not\models A(t).$$

But the $A(t) \in \mathcal{B}$ are all instances $\exists x A(x)$, hence

$$([T, I], v) \not\models \exists x A(x).$$

HOMEWORK

1. Give an example of 4 formulas with finite or infinite proof search trees (decomposition trees). 2. Construct counter-models for the formulas in 1. Do it in two ways: find your own structure, follow the above examples, i.e. give a counter-model determined by the proof search tree.

2. Write the proof of completeness theorem for **QRS**. I.e. Follow the above examples to show the implication:

$$If \quad \not\vdash A \;\; then \quad \not\models A$$

for ANY formula A.