1 Completeness Theorem for First Order Logic

There are many proofs of the Completeness Theorem for First Order Logic. We follow here a version of Henkin's proof, as presented in the *Handbook of Mathematical Logic*. It contains a method for reducing certain problems of first-order logic back to problems about propositional logic. We give independent proof of Compactness Theorem for propositional logic. The Compactness Theorem for first-order logic and Löwenheim-Skolem Theorems and the Gödel Completeness Theorem fall out of the Henkin method.

1.1 Compactness Theorem for Propositional Logic

Let $\mathcal{L} = \mathcal{L}(\mathbf{P}, \mathbf{F}, \mathbf{C})$ be a first order language with equality. We assume that the sets **P, F, C** are infinitely enumerable. We define a **propositional logic** within it as follows.

Prime formulas We consider a subset P of the set $\mathcal{F}$ of all formulas of $\mathcal{L}$. Intuitively these are formulas of $\mathcal{L}$ which are not direct propositional combination of simpler formulas, that is, *atomic formulas* ($A\mathcal{F}$) and formulas beginning with quantifiers.

Formally, we have that

$P = \{A \in \mathcal{F} : \quad A \in A\mathcal{F} \quad or \quad A = \forall x B, \quad A = \exists x B \quad for \quad B \in \mathcal{F}\}.$

Example 1.1 *The following are primitive formulas.*

$R(t_1, t_2), \quad \forall x(A(x) \Rightarrow \neg A(x)), \quad (c = c), \quad \exists x(Q(x, y) \cap \forall y A(y)).$

The following are not primitive formulas.

$(R(t_1, t_2) \Rightarrow (c = c)), \quad (R(t_1, t_2) \cup \forall x(A(x) \Rightarrow \neg A(x)).$

Given a set P of primitive formulas we define in a standard way the set $P\mathcal{F}$ of *propositional formulas* as follows.

Propositional formulas The smallest set $P\mathcal{F} \subset \mathcal{F}$ such that

1. $P \subset P\mathcal{F}$
2. If $A, B \in P\mathcal{F}$, then $(A \Rightarrow B), (A \cup B), (A \cap B)$, and $\neg A \in P\mathcal{F}$

is called a set of propositional formulas of the first order language $\mathcal{L}$.

We define propositional semantics for propositional formulas in $P\mathcal{F}$ as follows.

Truth assignment Let P be a set of prime formulas and $\{T, F\}$ be a two element set, thought as the set of logical values "true" and "false". Any function

$$v : P \longrightarrow \{T, F\}$$

is called *truth assignment* (or variable assignment).

Let $\mathbf{B} = (\{T, F\}, \Rightarrow, \cup, \cap, \neg)$ be a two-element Boolean algebra and $\mathbf{PF} = (P\mathcal{F}, \Rightarrow, \cup, \cap, \neg)$ a similar algebra of propositional formulas.

We extend v to a homomorphism

$v^* : \mathbf{PF} \longrightarrow \mathbf{B}$

in a usual way, i.e. we put $v^*(A) = v(A)$ for $A \in P$, and for any $A, B \in P\mathcal{F}$,

$v^*(A \Rightarrow B) = v^*(A) \Rightarrow v^*(B)$,

$v^*(A \cup B) = v^*(A) \cup v^*(B)$,

$v^*(A \cap B) = v^*(A) \cap v^*(B)$,

$v^*(\neg A) = \neg v^*(A)$.

Propositional Model A truth assignment v is called a *propositional model* for a formula $A \in P\mathcal{F}$ iff $v^*(A) = T$.

Propositional Tautology A formula $A \in P\mathcal{F}$ is a *propositional tautology* if $v^*(A) = T$ for all $v : P \longrightarrow \{T, F\}$.

For the sake of simplicity we will often say *model, tautology* instead *propositional model, propositional tautology.*

Model for the Set Given a set S of propositional formulas. We say that v is a *model for the set* S if v is a model for all formulas $A \in S$.

Consistent Set A set S of propositional formulas is *consistent* (in a sense of propositional logic) if it has a (propositional) model.

Theorem 1.1 (Compactness Theorem for Propositional Logic) *A set S of propositional formulas is* consistent *if and only if every finite subset of S is consistent.*

proof If S is a consistent set, then its model is also a model for all its finite subsets and all its finite subsets are consistent.

We prove the nontrivial half of the Compactness Theorem in a slightly modified form. To do so, we introduce the following definition.

Finitely Consistent Set (FC) Any set S such that all its subsets are consistent is called finitely consistent.

We use this definition to re-write the Compactness Theorem as: *A set S of propositional formulas is* consistent *if and only it is* finitely consistent. The nontrivial half of it is:

Every finitely consistent set of propositional formulas is consistent.

The proof of the nontrivial half of the Compactness Theorem, as stated above, consistes of the following four steps.

Step 1 We introduce the notion of a *maximal finitely consistent set.*

Step 2 We show that every *maximal finitely consistent set* is consistent by constructing its model.

Step 3 We show that every *finitely consistent set S* can be extended to a *maximal finitely consistent set S^**. I.e we show that for every finitely consistent set S there is a set S^*, such that $S \subset S^*$ and S^* is maximal finitely consistent.

Step 4 We use steps 2 and 3 to justify the following reasoning. Given a *finitely consistent* set S. We extend it, via construction defined in the step 2 to a *maximal finitely consistent* set S^*. By the step 2, S^* is consistent and hence so is the set S, what ends the proof.

Step 1: Maximal Finitely Consistent Set We call S *maximal finitely consistent* if S is finitely consistenst and for every formula A, either $A \in S$.

We use notation MFC for maximal finitely consistent set, and FC for the finitely consistent set.

Step 2: Any MFC set is consistent Given a MFC set S^*, we prove its consistency by constructing a truth assignment $v: \ P \ \longrightarrow \ \{T, F\}$ such that for all $A \in S^*$, $v^*(A) = T$.

Observe that the MFC sets have the following property.

MCF Property *For any MFC set S^*, for every $A \in P\mathcal{F}$, exactly one of the formulas $A \neg A$ belongs to S^*.*

In particular, for any $P \in P\mathcal{F}$, we have that exactly one of $P, \neg P \in S^*$. This justify the correctness of the following definition.

Let $v: \ P \ \longrightarrow \ \{T, F\}$ be a mapping such that

$$v(P) = \begin{cases} T & \text{if } P \in S^* \\ F & \text{if } P \notin S^* \end{cases}$$

We extend v to $v^* : \mathbf{PF} \longrightarrow \mathbf{B}$ in a usual way. In order to prove that v is a *model* for S^* we have to show that for any $A \in P\mathcal{F}$,

$$v^*(A) = \begin{cases} T & \text{if } A \in S^* \\ F & \text{if } A \notin S^* \end{cases}$$

We prove it by induction on the degree of the formula A. The base case of $A \in P$ follows immediately from the definition of v.

Case $A = \neg C$ Assume that $A \in S^*$. This means $\neg C \in S^*$ and by **MCF Property** we have that $C \notin S^*$. So by the inductive assumption $v^*(C) = F$ and $v^*(A) = v^*(\neg C) = \neg v^*(C) = \neg F = T$.

Assume now that $A \notin S^*$. By **MCF Property** we have that $C \in S^*$. By the inductive assumption $v^*(C) = T$ and $v^*(A) = v^*(\neg C) = \neg v^*(T) = \neg T = F$.

This proves that for any formula A,

$$v^*(\neg A) = \begin{cases} T & \text{if } \neg A \in S^* \\ F & \text{if } \neg A \notin S^* \end{cases}$$

Case $A = (B \cup C)$ Let $(B \cup C) \in S^*$. It is enough to prove that in this case $B \in S^*$ and $C \in S^*$, because then from the inductive assumption $v^*(C) = v^*(D) = T$ and $v^*(B \cup C) = v^*(B) \cup v^*(C) = T \cup T = T$.

Assume that $(B \cup C) \in S^*$, $B \notin S^*$ and $C \notin S^*$. Then by **MCF Property** we have that $\neg B \in S^*$, $\neg C \in S^*$ and consequently the set

$$\{(B \cup C), \neg B, \neg C\}$$

is a finite inconsistent subset of S^*, what contradicts the fact that S^* is finitely consistent.

Assume now that $(B \cup C) \notin S^*$. By **MCF Property**, $\neg(B \cup C) \in S^*$ and by the $A = \neg C$ we have that $v^*(\neg(B \cup C)) = T$. But $v^*(\neg(B \cup C)) = \neg v^*((B \cup C)) = T$ means that $v^*((B \cup C)) = F$, what end the proof of this case.

The remaining cases of $A = (B \cap C), A = (B \Rightarrow C)$ are similar to the above and are left to the reader as an exercise.

Step 3: Maximal finitely consistent extention Given a finitely consistent set S, we construct its *maximal finitely consistent extension* S^* as follows.

The set of all formulas of $\mathcal{L}$ is countable, so is $P\mathcal{F}$. We assume that all propositional formulas form a one-to-one sequence

$$A_1, A_2, ..., A_n, ... \tag{1}$$

We define a chain

$$S_0 \subset S_1 \subset S_2.... \subset S_n \subset .... \tag{2}$$

of *extentions* of the set S by

$$S_0 = S;$$

$$S_{n+1} = \begin{cases} S_n \cup \{A_n\} & \text{if } S_n \cup \{A_n\} \text{ is finitely consistent} \\ S_n \cup \{\neg A_n\} & \text{otherwise.} \end{cases}$$

We take

$$S^* = \bigcup_{n \in N} S_n. \tag{3}$$

Clearly, $S \subset S^*$ and for every A, either $A \in S^*$ or $\neg A \in S^*$. To finish the proof that S^* is MCF we have to show that it is finitely consistent.

First, let observe that if all sets S_n are finitely consistent, so is $S^* = \bigcup_{n \in N} S_n$. Namely, let $S_F = \{B_1, ..., B_k\}$ be a finite subset of S^*. This means that there are sets $S_{i_1}, ...S_{i_k}$ in the chain (2) such that $B_m \in S_{i_m}$, $m = 1, ..k$. Let $M = max(i_1, ...i_k)$. Obviously $S_F \subset S_M$ and S_M is finitely consistent as an element of the chain (2). This proves the if all sets S_n are finitely consistent, so is S^*.

Now we have to prove only that all S_n in the chain (2) are finitely consistent. We carry the proof by induction over the length of the chain. $S_0 = S$, so it is FC by assumption of the Compactness Theorem. Assume now that S_n is FC, we prove that so is S_{n+1}. We have two cases to consider.

Case 1 $S_{n+1} = S_n \cup \{A_n\}$, then S_{n+1} is FC by the definition of the chain (2).

Case 2 $S_{n+1} = S_n \cup \{\neg A_n\}$. Observe that this can happen only if $S_n \cup \{A_n\}$ is not FC, i.e. there is a finite subset $S_n^{'} \subset S_n$, such that $S_n^{'} \cup \{A_n\}$ is not consistent.

Suppose now that S_{n+1} is not FC. This means that there is a finite subset $S_n^{''} \subset S_n$, such that $S_n^{''} \cup \{\neg A_n\}$ is not consistent.

Take $S_n^{'} \cup S_n^{''}$. It is a finite subset of S_n so is consistent by the inductive assumption. Let v be a model of $S_n^{'} \cup S_n^{''}$. Then *one* of $v^*(A), v^*(\neg A)$ *must be* T. This contradicts the inconsistency of both $S_n^{'} \cup \{A_n\}$ and $S_n^{'} \cup \{\neg A_n\}$.

Thus, in ether case, S_{n+1}, is after all consistent. This ends the proof of the Step 3 and of the Compactness Theorem via the argument presented in the Step 4.

1.2 Reduction of first-order logic to propositional logic

Propositional tautologies as defined in the previous section barely scratch the surface of the collection of first -order tautologies, or first order *valid* formulas, as they are often called. For example the following first-order formulas are propositional tautologies,

$$(\exists x A(x) \cup \neg \exists x A(x)),$$
$$(\forall x A(x) \cup \neg \forall x A(x)),$$
$$(\neg(\exists x A(x) \cup \forall x A(x)) \Rightarrow (\neg \exists x A(x) \cap \neg \forall x A(x))),$$

but the following are first order tautologies (valid formulas) that are not propositional tautologies:

$$\forall x(A(x) \cup \neg A(x)),$$
$$(\neg \forall x A(x) \Rightarrow \exists x \neg A(x)).$$

The first formula above is just a prime formula, the second is of the form $(\neg B \Rightarrow C)$, for B and C prime.

To stress the difference between the propositional and first order tautologies some books reserve the word *tautology* for the propositional tautologies alone, using the notion of *valid formula* for the first order tautologies. We use here both notions, with the preference to *first-order tautology* or *tautology* for short when there is no room for misunderstanding.

To make sure that there is no misunderstandings we remind the following definitions.

Given a first order language $\mathcal{L}$ with the set of variables VAR and the set of formulas $\mathcal{F}$. Let $\mathcal{M} = [M, I]$ be a structure for the language $\mathcal{L}$, with the universe M and the interpretation I and let $s : VAR \longrightarrow M$ be a valuation of $\mathcal{L}$ in M.

A is true in $\mathcal{M}$ Given a structure $\mathcal{M} = [M, I]$, we say that a formula A is true in $\mathcal{M}$ if there is a valuation $s : VAR \longrightarrow M$ such that

$$(\mathcal{M}, s) \models A.$$

A is valid in $\mathcal{M}$ Given a structure $\mathcal{M} = [M, I]$, we say that a formula A is valid in $\mathcal{M}$ if

$$(\mathcal{M}, s) \models A$$

for all valuations $s : VAR \longrightarrow M$.

Model $\mathcal{M}$ If A is valid in a structure $\mathcal{M} = [M, I]$, then $\mathcal{M}$ is called a model of A.

A is valid A formula A called is valid if it is valid in all structures $\mathcal{M} = [M, I]$, i.e. if all structures are models of A.

A is a first-order tautology A valid formula A is also called a first-order tautology, or tautology, for short.

Case: A is a sentence If A is a sentence, then the truth or falsity of $(\mathcal{M}, s) \models A$ is completely independent of s. Thus we write

$$\mathcal{M} \models A$$

and read $\mathcal{M}$ *is a model of* A, if for some (hence every) valuation s, $(\mathcal{M}, s) \models A$.

Model of a set of sentences $\mathcal{M}$ is a model of a set S of sentences if $\mathcal{M} \models A$ for all $A \in S$. We write it

$$\mathcal{M} \models S.$$