1 Completeness Theorem for Classical Predicate Logic

The relationship between the first order models defined in terms of structures $\mathcal{M} = [M, I]$ and valuations $s : VAR \longrightarrow M$ and propositional models defined in terms of truth assignments $v : P \longrightarrow \{T, F\}$ is established by the following lemma.

Lemma 1.1 (Predicate and Propositional Models)

Let $\mathcal{M} = [M, I]$ be a structure for the language $\mathcal{L}$ and let $s : VAR \longrightarrow M$ a valuation in $\mathcal{M}$. There is a truth assignments $v : P \longrightarrow \{T, F\}$ such that for all formulas A of $\mathcal{L}$,

$$(\mathcal{M}, s) \models A \quad if \ and \ only \ if \quad v^*(A) = T.$$

In particular, for any set S of sentences of $\mathcal{L}$,

if $\mathcal{M} \models S$ then S is consistent in sense of propositional logic.

Proof For any prime formula $A \in P$ we define

$$v(A) = \begin{cases} T & \text{if } (\mathcal{M}, s) \models A \\ F & \text{otherwise.} \end{cases}$$

Since every formula in $\mathcal{L}$ is either prime or is built up from prime formulas by means of propositional connectives, the conclusion is obvious.

Observe, that the converse of the lemma is far from true. Consider a set

$$S = \{\forall x(A(x) \Rightarrow B(x)), \forall x A(x), \exists x \neg B(x)\}.$$

All formulas of S are different prime formulas, S is hence consistent in the sense of propositional logic and obviously has no (predicate) model.

The language $\mathcal{L}$ is a predicate language with equality. We adopt a following set of axioms.

Equality Axioms For any free variable or constant of $\mathcal{L}$, i.e for any $u, w, u_i, w_i \in (VAR \cup \mathbf{C})$,

E1 $u = u$,

E2 $(u = w \Rightarrow w = u)$,

E3 $((u_1 = u_2 \cap u_2 = u_3) \Rightarrow u_1 = u_3)$,

E4 $((u_1 = w_1 \cap ... \cap u_n = w_n) \Rightarrow (R(u_1, ..., u_n) \Rightarrow R(w_1, ..., w_n)))$,

E5 $((u_1 = w_1 \cap ... \cap u_n = w_n) \Rightarrow (t(u_1, ..., u_n) \Rightarrow t(w_1, ..., w_n)))$,

where $R \in bfP$ and $t \in T$, i.e. R is an arbitrary n-ary relation symbol of $\mathcal{L}$ and t is an arbitrary n-ary term of $\mathcal{L}$.

Obviously, all equality axioms are first-order*tautologies*, or are *valid* formulas of $\mathcal{L}$, i.e. for all $\mathcal{M} = [M, I]$ and all $s : VAR \longrightarrow M$, and for all $A \in \{E1, E2, E3, E4, E5.E6\}, (\mathcal{M}, s) \models A$.

This is why we still call logic with equality axioms added a logic.

Now we are going to define notions that is fundamental to the Henkin's technique for reducing first-order logic to propositional logic. The first one is that of *witnessing expansion* of the language $\mathcal{L}$.

Witnessing expansion $\mathcal{L}(C)$ of $\mathcal{L}$ We construct an expansion of our language $\mathcal{L}$ by adding a set C of new constants to it, i.e. we define a new language $\mathcal{L}(C)$

$$\mathcal{L}(C) = \mathcal{L}(\mathbf{P}, \mathbf{F}, (\mathbf{C} \cup C))$$

which is usually denoted shortly as

$$\mathcal{L}(C) = \mathcal{L} \cup C.$$

Definition of C We define the set C of new constants by constructing an infinite sequence

$$C_0, C_1, ..., C_n, ... \tag{1}$$

of sets of constants together with an infinite sequence

$$\mathcal{L}_0, \mathcal{L}_1, ..., \mathcal{L}_n, ... \tag{2}$$

of languages, such that

$$\mathcal{L}_n = \mathcal{L} \cup C_n, \quad C = \bigcup_{n \in N} C_n$$

and

$$\mathcal{L}(C) = \mathcal{L} \cup C.$$

We define sequences (1), (2) as follows. Let

$$C_0 = \emptyset, \quad \mathcal{L}_0 = \mathcal{L} \cup C_0 = \mathcal{L}.$$

We denote by

$$A[x]$$

the fact that the formula A has exactly one free variable and for each such a formula we introduce a distinct new constant denoted by

$$c_{A[x]}.$$

We define

$$C_1 = \{c_{A[x]} : \quad A[x] \ \in \mathcal{L}_0\}, \quad \mathcal{L}_1 = \mathcal{L} \cup C_1.$$

Assume that we have defined C_n and $\mathcal{L}_n$. We introduce a distinct new constant $c_{A[x]}$ for each formula $A[x]$ of $\mathcal{L}_n$ which is not already a formula of $\mathcal{L}_{n-1}$ (i.e., if some constant from C_n appears in $A[x]$). We write it informally as $A[x] \in (\mathcal{L}_n - \mathcal{L}_{n-1})$.

We define

$$C_{n+1} = C_n \cup \{c_{A[x]} : \quad A[x] \in (\mathcal{L}_n - \mathcal{L}_{n-1})\},$$
$$\mathcal{L}_{n+1} = \mathcal{L} \cup C_{n+1}.$$

Witnessing constant For any formula A, a constant $c_{A[x]}$ as defined above is called a *witnessing constant.*

Henkin Axioms The following sentences

H1 $(\exists x A(x) \Rightarrow A(c_{A[x]}))$,

H2 $(A(c_{\neg A[x]}) \Rightarrow \forall x A(x))$

are called Henkin axioms.

The informal idea behind the Henkin axioms is the following.

The axiom H1 says:
If $\exists x A(x)$ *is true in a structure, choose an element a satisfying* $A(x)$ *and give it a new name* $c_{A[x]}$.

The axiom H2 says:
If $\forall x A(x)$ *is false, choose a counterexample b and call it by a new name* $c_{\neg A[x]}$.

Quantifier axioms The following sentences

Q1 $(\forall x A(x) \Rightarrow A(t))$, t is a closed term of $\mathcal{L}(C)$;

Q2 $(A(t) \Rightarrow \exists x A(x))$, t is a closed term of $\mathcal{L}(C)$

are called *quantifier axioms.* They obviously are first-order tautologies.

Henkin set Any set of sentences of $\mathcal{L}(C)$ which are either *Henkin axioms* H1, H2 or quantifier axioms Q1, Q2 is called *Henkin set* and denoted by

$$S_{Henkin}.$$

The set S_{Henkin} is obviously not true in every $\mathcal{L}(C)$-structure, but we are going to show that every $\mathcal{L}$ -structute can be turned into an $\mathcal{L}(C)$-structure which is *model* of S_{Henkin}. Before we do so we need to introduce two new notions.

Reduct and Expension Given two languages $\mathcal{L}$ and $\updownarrow^{'}$ such that $\mathcal{L} \subset \mathcal{L}^{'}$. Let $\mathcal{M}^{'} = [M, I^{'}]$ be a structure for $\mathcal{L}^{'}$. The structure

$$\mathcal{M} = [M, I^{'} \mid \mathcal{L}]$$

is called the *reduct* of $\mathcal{M}^{'}$ to the language $\mathcal{L}$ and $\mathcal{M}^{'}$ is called the *expension* of $\mathcal{M}$ to the language $\mathcal{L}^{'}$.

Thus the reduct and the expension $\mathcal{M}^{'}$ and $\mathcal{M}$ are the same except that $\mathcal{M}^{'}$ assigns meanings to the symbols in $(\mathcal{L} - \mathcal{L}^{'})$.

Lemma 1.2 *Let $\mathcal{M} = [M, I]$ be any structure for the language $\mathcal{L}$ and let $\mathcal{L}(C)$ be the witnessing expansion of $\mathcal{L}$. There is an extention $\mathcal{M}^{'} = [M, I^{'}]$ of $\mathcal{M} = [M, I]$ such that $\mathcal{M}^{'}$ is a model of the set S_{Henkin}.*

Proof In order to define the expansion of $\mathcal{M}$ to $\mathcal{M}^{'}$ we have to define the interpretation $I^{'}$ for the symbols of the language $\mathcal{L}(C) = \mathcal{L} \cup C$, such that $I^{'} \mid \mathcal{L} = I$. This means that we have to define $c_{I^{'}}$ for all $c \in C$. By the definition, $c_{I^{'}} \in M$, so this also means that we have to assign the elements of M to all constants $c \in C$ in such a way that the resulting expansion is a model for all sentences from S_{Henkin}.

The quantifier axioms $Q1, Q2$ are first order tautologies so they are going to be true regardless, so we have to worry only about the Henkin axioms $H1, H2$. Observe now that if the lemma holds for the Henkin axioms H1, then it must hold for the axioms $H2$. Namely, let's consider the axiom $H2$:

$$(A(c_{\neg A[x]}) \Rightarrow \forall x A(x)).$$

Assume that $A(c_{\neg A[x]})$ is true in the expansion $\mathcal{M}^{'}$, ie. that $\mathcal{M}^{'} \models A(c_{\neg A[x]})$ and that $\mathcal{M}^{'} \not\models \forall x A(x)$. This means that $\mathcal{M}^{'} \models \neg \forall x A(x)$ and by the de Morgan Laws, $\mathcal{M}^{'} \models \exists x \neg A(x)$. But we have assumed that $\mathcal{M}^{'}$ is a model for $H1$. In particular $\mathcal{M}^{'} \models (\exists x \neg A(x) \Rightarrow \neg A(c_{\neg A[x]}))$, and hence $\mathcal{M}^{'} \models \neg A(c_{\neg A[x]})$ and this contradicts the assumption that $\mathcal{M}^{'} \models A(c_{\neg A[x]})$. Thus if $\mathcal{M}^{'}$ is a model for all axioms of the type $H1$, it is also a model for all axioms of the type $H2$.

We define $c_{I'}$ for all $c \in C = \bigcup C_n$ by induction on n. Let $n = 1$ and $c_{A[x]} \in C_1$. By definition, $C_1 = \{c_{A[x]} : A[x] \in \mathcal{L}\}$. In this case we have that $\exists x A(x) \in \mathcal{L}$ and hence the notion $\mathcal{M} \models \exists x A(x)$ is well defined, as $\mathcal{M} = [M, I]$ is the structure for the language $\mathcal{L}$. As we consider arbitrary structure $\mathcal{M}$, there are two posibilities: $\mathcal{M} \models \exists x A(x)$ or $\mathcal{M} \not\models \exists x A(x)$. We define $c_{I'}$, for all $c \in C_1$ as follows.

If $\mathcal{M} \models \exists x A(x)$, then $(\mathcal{M}, v') \models A(x)$ for certain $v'(x) = a \in M$. We set $(c_{A[x]})_{I'} = a$. If $\mathcal{M} \not\models \exists x A(x)$, we set $(c_{A[x]})_{I'}$ arbitrarily. Obviously, $\mathcal{M}^{'} = (M, I^{'}) \models (\exists x A(x) \Rightarrow A(c_{A[x]}))$. But once $c \in C_1$ are all interpreted in $\mathcal{M}^{'} = (M, I^{'})$, then the notion $\mathcal{M}^{'} \models A$ is defined for all formulas $A \in \mathcal{L} \cup C_1$. We carry the inductive step in the exactly the same way as the one above.

Canonical structure Given a structure $\mathcal{M} = [M, I]$ for the language $\mathcal{L}$. The extention $\mathcal{M}^{'} = [M, I^{'}]$ of $\mathcal{M} = [M, I]$ is called a *canonical structure for* $\mathcal{L}(C)$ if all $a \in M$ are denoted by some $c \in C$, i.e if

$$M = \{c_{I'} : \ c \in C\}.$$

Now we are ready to state and proof a lemma that provides the essential step in the proof of the Completeness Theorem.

Lemma 1.3 (The reduction to propositional logic) *Let $\mathcal{L}$ be a first order language and let $\mathcal{L}(C)$ be a witnessing expansion of $\mathcal{L}$. For any set S of* sentences *of $\mathcal{L}$ the following conditions are equivalent.*

(i) *S has a model, i.e. there is a structure $\mathcal{M} = [M, I]$ for the language $\mathcal{L}$ such that $\mathcal{M} \models A$ for all $A \in S$.*

(ii) *There is a* canonical $\mathcal{L}(C)$ structure $\mathcal{M}^{'} = [M, I^{'}]$ *which is a model for S, i.e. such that $\mathcal{M}^{'} \models A$ for all $A \in S$.*

(iii) *The set $S \cup S_{Henkin} \cup EQ$ is consistent in sense of propositional logic, where EQ denotes the equality axioms $E1 - E5$.*

Proof The implication $(ii) \rightarrow (i)$ is immediate. The implication $(i) \rightarrow (iii)$ follows from lemma 1.2. We have to prove only the implication $(iii) \rightarrow (ii)$.

Assume that the set $S \cup S_{Henkin} \cup EQ$ is consistent in sense of propositional logic and let v be a truth assignment to the prime sentences of $\mathcal{L}(C)$, such that $v^*(A) = T$ for all $A \in S \cup S_{Henkin} \cup EQ$. To prove the lemma, we construct a canonical model $\mathcal{M}^{'} = [M, I^{'}]$ such that, for all sentences A of $\mathcal{L}(C)$,

$$\mathcal{M}^{'} \models \ A \ \ if \ and \ only \ if \ v^*(A) = T.$$

v is a propositional model for the set S_{Henkin}, so v^* satisfies the following conditions:

$$v^*(\exists x A(x)) = T \ \ if\ and\ only\ if \ \ v^*(A(c_{A[x]})) = T, \tag{3}$$

$$v^*(\forall x A(x)) = T \ \ if\ and\ only\ if \ \ v^*(A(t)) = T, \tag{4}$$

for all closed terms t of $\mathcal{L}(C)$.

The conditions (3) and (4) allow us to construct the model $\mathcal{M}' = [M, I']$ out of the constants in C in the following way.

TO BE DONE!

The Main Lemma provides not only a method of constructing models of theories out of symbols, but also gives us immediate proofs of the Compactness Theorem for the first order logic and Lowenheim-Skolem Theorem.

Theorem 1.1 (Compactness theorem for the first order logic)
Let S be any set of first order formulas. The set S has a model if and only if any finite subset S_0 of S has a model.

Proof Let S be a set of first order formulas such that every finite subset S_0 of S has a model. We need to show that S has a model. By the implication $(iii) \rightarrow (i)$ of the Main Lemma 1.3 this is equivalent to proving that $S \cup S_{Henkin} \cup EQ$ is consistent in the sense of propositional logic. By the Compactness Theorem **??** for propositional logic, it suffices to prove that for every finite subset $S_0 \subset S$, $S_0 \cup S_{Henkin} \cup EQ$ is consistent, which follows from the hypothesis and the implication $(i) \rightarrow (iii)$ of the Main Lemma 1.3.

Theorem 1.2 (Löwenheim-Skolem Theorem)
Let κ be an infinite cardinal and let S be a set of at most κ formulas of the first order language. If the set S has a model, then there is a model $\mathcal{M} = [M, I]$ of S such that $card M \leq \kappa$.

Proof Let $\mathcal{L}$ be a first order language with the alphabet $\mathcal{A}$ such that $card(\mathcal{A}) \leq \kappa$. Obviously, $card(\mathcal{F}) \leq \kappa$. By the definition of the witnessing expansion $\mathcal{L}(C)$ of $\mathcal{L}$, $C = \bigcup_n C_n$ and for each n, $card(C_n) \leq \kappa$. So also $card C \leq \kappa$. Thus any canonical structure for $\mathcal{L}(C)$ has $\leq \kappa$ elements. By the implication $(i) \rightarrow (ii)$ of the Main Lemma 1.3 there is a model of S (canonical structure) with $\leq \kappa$ elements.