CHAPTER 2

INTRODUCTION TO CLASSICAL PROPOSITIONAL LOGIC

1 Motivation and History

The origins of the classical propositional logic, classical propositional calculus, as it was, and still often is called, go back to antiquity and are due to Stoic school of philosophy (3rd century B.C.), whose most eminent representative was Chryssipus. But the real development of this calculus began only in the mid-19th century and was initiated by the research done by the English mathematician G. Boole, who is sometimes regarded as the founder of mathematical logic. The classical propositional calculus was first formulated as a formal *axiomatic system* by the eminent German logician G. Frege in 1879.

The assumption underlying the formalization of classical propositional calculus are the following.

Logical sentences We deal only with sentences that can always be evaluated as *true* or *false*. Such sentences are called *logical sentences* or *propositions*. Hence the name *propositional logic*.

A statement: $2+2=4$ is a proposition as we assume that it is a well known and agreed upon truth. A statement: $2+2=5$ is also a proposition (false. A statement:] *I am pretty* is modeled, if needed as a logical sentence (proposition). We assume that it is false, or true. A statement: $2+n=5$ **is not** a proposition; it might be true for some n, for example n=3, false for other n, for example n= 2, and moreover, we don't know what n is. Sentences of this kind are called **propositional functions**. We model propositional functions within propositional logic by treating propositional functions as propositions.

The similar examples can be found in natural language rather then in mathematical language. For example we tend to accept a statement: *The earth circulates around the sun* as a logical sentence while a statement: *Ann is pretty*, even if can be modeled as a proposition by assuming that is always true or false, could also be treated as ambiguous; Ann may be found pretty by some people and not pretty by others. If we try to improve the situation by saying for example: *Ann seems to be pretty*, *I am sure Ann is pretty* or even *I know that Ann is pretty* the ambiguity increases rather then decreases. To deal with those ambiguities many non-classical logics were and are being invented and examined by philosophers, computer scientists, and even by mathematicians. We will present and study some of them later. Nevertheless we accept these statements in classical logic, but we treat them (model them) simply as true or false.

The classical logic reflects the "black" and "white" qualities of mathematics; we expect from mathematical theorems to be always either true or false and the reasonings leading to them should guarantee this without any ambiguity.

Logical sentences may be combined in various ways to form more complicated sentences, called **formulas**. We combine them using the following words or phrases:

not; *and*; *or*; *if ..., then*; *if and only if.*

Moreover, we use **symbols** do denote both *logical sentences* and the above phrases, hence the name **symbolic logic**.

We use **symbols** $a, b, c, p, r, q, ..$ for logical sentences. We use a symbol $\neg$ for "*not*", a a symbol $\cap$ for "*and*", a symbol $\cup$ for "*or*", a symbol $\Rightarrow$ for *"if ..., then"*, and a symbol $\Leftrightarrow$ for *"if and only if"*.

Example 1

To translate a natural language sentence:

The fact that it is not true that at the same time $2+2=4$ *and* $2+2=5$ *implies that* $2+2=4$

into its **symbolic logic** form we write it first in a form:

If *not* $(2+2=4$ *and* $2+2=5)$ *then* $2+2=4$

and then in a symbolic form:

$(\neg(a \cap b) \Rightarrow a)$.

The formal description of the symbols we use and the way we construct the formulas is called *a syntax* of the *symbolic (or formal) language* of the classical propositional calculus. It is defined intuitively below, and it will be defined in full formality in the next chapter.

2 Propositional Language

We use the name *symbolic logic* to stress the fact that it deals with *symbols* only. The syntax establishes the set of symbols, called *the alphabet*, and describes precisely how to form acceptable expressions, called *formulas.* Such established syntax is also called *a formal language* of a given logic. There are *propositional languages* which are defined to describe propositional logics and *predicate languages* to describe more complex logics, called predicate logics or predicate calculi. They are also called *zero order* and *first order* languages (and logics), respectively. We will use here the terms *propositional* and *predicate* languages and logics.

The formal language symbols do not carry with them any *logical value.* We assign a logical value to them in a separate step. This step is called *a semantics.* We will see later that a given language can have many different semantics. Different semantics will define different logics.

The classical propositional semantics is described in the next section.

Propositional language consists of **a propositional alphabet** and **a set of formulas** (propositional).

Propositional Alphabet consists of an countably infinite set of *variables* and a finite set of *propositional connectives.*

Variables are the symbols denoting logical sentences (propositions) and hence are called **propositional variables**. We denote the propositional variables by letters *a, b, c,....,* with indices if necessary. I.e. we can also use $a_1, a_2, ..., b_1, b_2, ...$ etc... as symbols for propositional variables.

The symbols for connectives are: $\neg$, $\cap$, $\cup$, $\Rightarrow$, $\Leftrightarrow$ and they are called *a negation*, *a conjunction*, *a disjunction*, *an implication*,and *an equivalence*, respectively.

Formulas are expressions build by means of logical connectives and variables and are be denoted by $A, B, C, ...$, with indices, if necessary. They are defined recursively as follows.

Base step : The propositional variables are formulas and are called **atomic formulas**.

Recursive step: if we already have two formulas A, B, then we adopt the expression: $(A \cap B), (A \cup B), (A \Rightarrow B), (A \Leftrightarrow B)$ and also $\neg A$ as formulas.

Example 1

By the definition, any propositional variable is a formula. Let's take, for example two variables a and b. They are atomic formulas.

By the recursive step we get that

$$(a \cap b), (a \cup b), (a \Rightarrow b), (a \Leftrightarrow b), \neg a, \neg b$$

are formulas.

Recursive step applied again produces for example formulas :

$$\neg(a \cap b), ((a \Leftrightarrow b) \cup \neg b), \neg\neg a, \neg\neg(a \cap b).$$

These are not all formulas we can obtain in the first recursive step.

Moreover, as the recursive process continue we obtain an countably infinite set of all formulas.

Remark that we put parenthesis within the formulas in a way to avoid **ambiguity.**

The expression: $a \cap b \cup a$, is ambiguous. We don't know whether it represents $(a \cap b) \cup a$ or $a \cap (b \cup a)$. So, it is not a formula.

3 Truth Tables Semantics

We present here definitions of propositional connectives in terms of two logical values (*true* or *false*) and discuss their motivations.

The resulting definitions are called *a semantics for the classical propositional connectives.* As we consider only two logical values, the semantics is also called 2 valued semantics. It is expressed in terms of *truth tables* for logical connectives, and hence is also called *a truth tables semantics.*

The semantics presented here is fairly informal. The formal definition of classical propositional semantics will be presented in chapter 4.

3.1 Classical Connectives

We assumed in section 2 that we our language contains 5 connectives called conjunction, disjunction, implication, equivalence, and negation. We define their semantics, i.e. their definitions in terms of two logical values as follows.

CONJUNCTION - Motivation and definition

As we have defined in section 2, the symbol $\cap$ is used instead of the word *and*, and is called the *symbol of conjunction.* The formula $(A \cap B)$ is called the *conjunction* of the formulas A and B, and A and B will be called *factors* of that conjunction.

In accordance with intuition, a conjunction $(A \cap B)$ is a *true* formula if both of its factors are *true* formulas. If one of the factors, or both, are *false* formulas, then the conjunction is a *false* formula.

Let's now denote *A is false* by $A = F$, what stands for *the logical value of A is F*, and *A is true* by $A = T$, what stands for *the logical value of A is T*. We can see that the logical value of a conjunction depends on the logical values of its factors in a way which can be express in the form of the following table.

Conjunction Table

A	B	$(A \cap B)$
T	T	T
T	F	F
F	T	F
F	F	F

(1)

DISJUNCTION - Motivation and definition.

The symbol $\cup$ is used instead of the word *or*, and is called the *symbol of disjunction.* The *formula* $(A \cup B)$ is called the *disjunction* of the formulas A and B, and A and B will be called *factors* of that disjunction.

In everyday language the word *or* is used in two different senses. In the first, a statement of the form A *or* B is accepted as true if at least one of the statements A and B is true; in the other, the compound statement is accepted as true if one of the statements A and B is true, and the other is false. In mathematics the word *or* is used in the former sense.

Hence, we adopt the convention that a *disjunction* $(A \cup B)$ is *true* if at least one of the formulas A and B is true. This convention is called a classical semantics for the disjunction and is expressed in the following table.

Disjunction Table

A	B	$(A \cup B)$
T	T	T
T	F	T
F	T	T
F	F	F

(2)

As in the case of the conjunction, the logical value of a disjunction depends only on the logical values of its factors.

IMPLICATION - Motivation and definition.

A statement of the form *if A, then B* is written in symbols as $(A \Rightarrow B)$ and is called an *implication.* A is called its *antecedent*, B is called the *consequent.*

The semantics of the implication needs some discussion. In everyday language the implication statement *if A, then B* is interpreted to mean that B can be *inferred* from A. This interpretation differs from that given to it in mathematics, and hence in classical semantics.

The following example will explain how the semantics (logical values assignment) of the statement *if A, then B* is understood in mathematics.

Consider the following **arithmetical theorem:**

For every natural number n,

$$if \quad 6 \quad DIVIDES \quad n, \quad then \quad 3 \quad DIVIDES \quad n. \tag{3}$$

The above implication 3 is **true for any natural number**, hence, in particular, for 2,3,6.

Thus the following propositions are **true**:

$$If \quad 6 \quad DIVIDES \quad 2, \quad then \quad 3 \quad DIVIDES \quad 2. \tag{4}$$

$$If \quad 6 \quad DIVIDES \quad 3, \quad then \quad 3 \quad DIVIDES \quad 3. \tag{5}$$

$$If \quad 6 \quad DIVIDES \quad 6, \quad then \quad 3 \quad DIVIDES \quad 6. \tag{6}$$

It follows from 4 that an implication $(A \Rightarrow B)$ in which both the *antecedent* A and the *consequent* B are *false* statements is interpreted as a **true** statement.

It follows from 5 that an implication $A \Rightarrow B$ in which *false antecedent* A and *true consequent* B is interpreted as a **true** statement.

Finally, it follows from 6 that an implication $A \Rightarrow B$ in which both the *antecedent* A and the *consequent* B are *true* statements is interpreted as a **true** statement.

Thus one case remains to be examined, namely that in which the *antecedent* of an implication is a *true* statement, and the *consequent* is a *false* statement.

For example consider the statement:

If 6 DIVIDES 12, then 6 DIVIDES 5.

In accordance with arithmetic of natural numbers, this statement is interpreted as **false**.

The above examples justifies adopting the following semantics of an implication $(A \Rightarrow B)$.

An implication $(A \Rightarrow B)$ is interpreted to be a *false* statement if and only if its *antecedent* A is a *true* statement and its *consequent* is a *false* statement.

In the remaining cases such an implication is interpreted as a *true* statement.

This semantics is expressed in the form of the following table.

Implication Table

A	B	$(A \Rightarrow B)$
T	T	T
T	F	F
F	T	T
F	F	T

(7)

EQUIVALENCE - Motivation and definition.

There is still one binary propositional connective left; *if and only if.* We shall use a symbol $\Leftrightarrow$ for it and call it an *equivalence symbol.* An equivalence $(A \Leftrightarrow B)$ is, in accordance with intuition, interpreted as *true* if both formulas A and B have the same logical value, that is, are either *both true* or *both false.* This is expressed in the following table.

Equivalence Table

A	B	$(A \Leftrightarrow B)$
T	T	T
T	F	F
F	T	F
F	F	T

(8)

NEGATION - Motivation and definition.

The symbol $\neg$ is adopted instead of the word *not* and is called *negation symbol.* The formula $\neg A$ is called the *negation* of the formula A.

In accordance with the intuition, the negation of a true formula is a false formula, and the negation of a false formula is a true formula. This is expressed in the following table.

Negation Table

A	$\neg A$
T	F
F	T

(9)

Extensional connectives are the connectives that have the following property:

the logical value of the formulas form by means of these connectives and certain given formulas depends only on the logical value(s) of the given formulas.

All classical connectives are extensional.

In our propositional languages we consider two categories of connectives: binary and unary.

Binary connectives are such connectives that they enable us to form a new formula from *two* formulas.

The classical connectives: $\cup, \cap, \Rightarrow,$ and $\Leftrightarrow$ are *binary propositional connectives.*

Unary connectives are such connectives that they enable us to form a new formula from *one* formula.

The classical connective $\neg$ is a *unary propositional connective.*

In everyday language there are expressions which are propositional connectives but are not extensional. They do not play any role in mathematics and so are not discussed in classical logic.

Other symbols (notations) for propositional connectives.

The symbols used in our book are not the only one used in mathematical, logical, or computer science literature.

Other symbols frequently employed for propositional connectives are listed in the table below:

Negation	Disjunction	Conjunction	Implication	Equivalence
$-$A	$(A \cup B)$	$(A \cap B)$	$(A \Rightarrow B)$	$(A \Leftrightarrow B)$
NA	DAB	CAB	IAB	EAB
$\overline{A}$	$(A \vee B)$	(A & B)	$(A \rightarrow B)$	$(A \leftrightarrow B)$
$\sim A$	$(A \vee B)$	$(A \cdot B)$	$(A \supset B)$	$(A \equiv B)$
A'	$(A + B)$	$(A \cdot B)$	$(A \rightarrow B)$	$(A \equiv B)$

The first of these systems of notation is the closest to ours and is drawn mainly from the algebra of sets and lattice theory. The second comes from the Polish logician *J. Łukasiewicz*. In this notation the binary connectives *precede* the propositional variables and are *not inserted* between them; this enables us to dispense with parenthesis; Łukasiewicz's notation is usually called the *Polish notation* and it is a *parenthesis-free notation.* The third was used by D. Hilbert. The fourth comes from Peano and Russell, while the fifth goes back to Schröder and Pierce.

3.2 Other Propositional Connectives

The propositional classical connectives $\cap, \cup, \Rightarrow, \Leftrightarrow, \neg$ defined in previous section are not the only extensional connectives. We define here all possible unary and binary two valued extensional connectives.

All Possible Unary Connectives
An extensional unary connective $\bigtriangledown$ enables us to form from any formula A, a new formula $\bigtriangledown A$, whose logical value is defined in terms of the logical value of A only, i.e. by means of a table of a type 9.

Thus there as many *unary connectives* as there are functions f from the set $\{T, F\}$ to the set $\{T, F\}$, that is $2^2 = 4$.

All Unary Connectives Table :

A	$\bigtriangledown_1 A$	$\bigtriangledown_2 A$	$\neg A$	$\bigtriangledown_4 A$
T	F	T	F	T
F	F	F	T	T

(10)

All Possible Binary Connectives

An extensional binary connective $\circ$ permits us to form, of any two formulas A and B, a new formula $(A \circ B)$, whose logical value is defined from the logical values A and Bonly, i.e. by means of a table similar to 1, 2, 7, 8.

So, there are as many *binary connectives* as many functions f from a set $\{T, F\} \times \{T, F\}$ (four elements) to a set $\{T, F\}$ (two elements) that is, $2^4 = 16$.

All Binary Connectives Table :

A	B	$(A \circ_1 B)$	$(A \cap B)$	$(A \circ_3 B)$	$(A \circ_4 B)$
T	T	F	T	F	F
T	F	F	F	T	F
F	T	F	F	F	T
F	F	F	F	F	F
A	B	$(A \downarrow B)$	$(A \circ_6 B)$	$(A \circ_7 B)$	$(A \leftrightarrow B)$
T	T	F	T	T	T
T	F	F	T	F	F
F	T	F	F	T	F
F	F	T	F	F	T
A	B	$(A \circ_9 B)$	$(A \circ_{10} B)$	$(A \circ_{11} B)$	$(A \cup B)$
T	T	F	F	F	T
T	F	T	T	F	T
F	T	T	F	T	T
F	F	F	T	T	F
A	B	$(A \circ_{13} B)$	$(A \Rightarrow B)$	$(A \uparrow B)$	$(A \circ_{16} B)$
T	T	T	T	F	T
T	F	T	F	T	T
F	T	F	T	T	T
F	F	T	T	T	T

(11)

3.3 Functional Dependency

It can be proved that all propositional connectives, as defined by tables 10 and 11, i.e. whether unary or binary, can be defined in terms of *disjunction* and *negation*.

This property of defining a set of connectives in terms of its proper subset is called **a functional dependency of connectives.**

There are also two other *binary* connectives which suffice, each of them separately, to define *all* connectives , whether unary or binary. These connectives play a special role and are denoted in our table 11 by ↓ and ↑, respectively.

The connective ↑ was discovered in 1913 by H.M. Sheffer, who called it *alternative negation*. Now it is often called a *Sheffer's connective*. The formula $A \uparrow B$ is read: *not both A and B*. As it is a special connective we re-write its truth table separately.

Sheffer's Connective Table :

A	B	$(A \uparrow B)$
T	T	F
T	F	T
F	T	T
F	F	T

(12)

Observe that $T \uparrow T = F$ and $F \uparrow F = T$. This means that logical value of a formula $(A \uparrow A)$ is the same as logical value of a formula $\neg A$, for any logical value the formula A can take. We write it as $\neg A = (A \uparrow A)$ and express it in a form of the table below.

The $\neg A = (A \uparrow A)$ **Table** :

A	B	$\neg A$	$(A \uparrow B)$
T	T	F	F
F	F	T	T

(13)

Definition of $\neg$ **in terms of** $\uparrow$.

We call the equality

$$\neg A = (A \uparrow A) \tag{14}$$

the definition of $\neg$ in terms of $\uparrow$. Its correctness is established by the table 13.

Observe now that the Sheffer's connective table 12 looks as a negation of the disjunction table 1. It means that the logical value a formula $(A \cap B)$ is the same as logical value of a formula $\neg(A \uparrow B)$, for all logical values of A and B. We write it as $(A \cap B) = \neg(A \uparrow B)$ and and express it in a form of the table below.

The $(A \cap B) = \neg(A \uparrow B)$ **Table** :

A	B	$(A \cap B)$	$\neg(A \uparrow B)$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

(15)

Definition of $\cap$ **in terms of** $\neg$ **and** $\uparrow$:

We call the equality

$$(A \cap B) = \neg(A \uparrow B) \tag{16}$$

the definition of conjunction in terms of negation and Sheffer's connective. Its correctness is established by the table 15.

Observe now that if we combine equations 14 and 16 and Tables 13 and 15 we get that $(A \cap B) = (A \uparrow B) \uparrow (A \uparrow B)$ for all logical values and the following table and definition, what we express in the following.

The $(A \cap B) = (A \uparrow B) \uparrow (A \uparrow B)$ **Table** :

A	B	$(A \cap B)$	$((A \uparrow B) \uparrow (A \uparrow B))$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

(17)

Definition of $\cap$ in terms of $\uparrow$:

We call the equality

$$(A \cap B) = (A \uparrow B) \uparrow (A \uparrow B) \tag{18}$$

the definition of conjunction in terms of Sheffer's connective alone. Its correctness is established by the table 17.

Exercise 1

Find an equality and a table defining $\cup$ in terms of $\uparrow$ alone.

Solution

Definition of $\cup$ in terms of $\uparrow$:

The equality

$$(A \cup B) = ((A \uparrow A) \uparrow (B \uparrow B)) \tag{19}$$

defines disjunction $\cup$ in terms of $\uparrow$. We check its correctness by constructing the following table.

A	B	$(A \cup B)$	$((A \uparrow A) \uparrow (B \uparrow B))$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

(20)

The connective $\downarrow$ was discovered and termed by J. Łukasiewics *joint negation.* The formula $A \downarrow B$ is read: *neither* A *nor* B. As it is a special connective we re-write its truth table separately.

Ł Connective Table :

A	B	$(A \downarrow B)$
T	T	F
T	F	F
F	T	F
F	F	T

(21)

Observe that $T \downarrow T = F$ and $F \downarrow F = T$. This means that logical value of a formula $(A \downarrow A)$ is the same as logical value of a formula $\neg A$, for any logical value the formula A can take. We write it as $A = (A \downarrow A)$ and express it in a form of the table below.

The $\neg A = (A \downarrow A)$ Table :

A	B	$\neg A$	$(A \downarrow B)$
T	T	F	F
F	F	T	T

(22)

Definition of $\neg$ in terms of $\downarrow$:

We call the equality

$$\neg A = (A \downarrow A) \tag{23}$$

the definition of negation in terms of Łukasiewicz's connective alone. Its correctness is established by the table 22.

Exercise 2

Prove that the equality

$$(A \cup B) = ((A \downarrow B) \downarrow (A \downarrow B)) \tag{24}$$

defines $\cup$ in terms of $\downarrow$.

Solution

Definition of $\cup$ in terms of $\downarrow$.

To prove the correctness of the equation 24 we construct a table below.

A	B	$(A \cup B)$	$((A \downarrow B) \downarrow (A \downarrow B))$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

(25)

It was proved in 1925 by E. Żyliński that no propositional connective other than ↑ and ↓ suffices to define all the remaining connectives.

4 Exercises and Homework Problems

The set first of problems presented here deals with "translation" of logical sentences into formulas and vice-versa, i.e. with giving examples of "real' sentences corresponding to a given formula.

Exercise 1

Given a sentence

If a natural number a is divisible by 3, then from the fact that a in not divisible by three we can deduce that a is divisible by 5

write a formula corresponding to this sentence.

Solution:

First we write our sentence in a more "logical form" as follows:

If a natural number a is divisible by 3, *then* (*if* *not*(a in divisible by three) *then* a is divisible by 5).

We assume that we can always decide, for any given natural number whether it or not divisible by 3 or 5 i.e. that the sentences involved here are *logical sentences.*

We denote the sentence:

"a natural number a is divisible by 3 " by a, and the sentence "a is divisible by 5" by b,

and we rewrite our sentence as:

If a, *then* (*if* *not* a, *then* b).

We replace expressions *If* ... *then* and *not* by symbols ⇒ and ¬, respectively and we follow the definition of the set of formulas to obtain a formula

$$(a \Rightarrow (\neg a \Rightarrow b))$$

which corresponds to our logical sentence.

Observe that for a given logical sentence there is only one schema of a logical formula corresponding to it. I.e. one can replace a by d and b by a_1 and get a formula $(d \Rightarrow (\neg d \Rightarrow a_1))$, or for example $(b \Rightarrow (\neg b \Rightarrow d_5))$.

We can, in fact, construct as many of those formulas as we wish, but all those formulas will have the same form as $(a \Rightarrow (\neg a \Rightarrow b))$. They will differ only on a choice of names for the propositional variables assigned corresponding to logical sentences.

The same happens, when we want to do the "inverse" transformation from a given formula A to a logical sentence corresponding to it. There may be as many of them as we can invent, but they all will be built in the same way; the way described by the formula A.

Exercise 2

Given a formula

$$(a \cap (\neg a \cup b))$$

write 2 sentences which correspond to this formula.

Solution:

Let propositional variables a, b denote sentences $2+2=4$ and $2>1$, respectively. In this case the corresponding sentence is:

$2+2=4$ *and we have that* $2+2 \neq 4$ *or* $2>1$.

If we assume that the propositional variables a, b denote sentences $2>1$ and $2+2=4$, respectively, then the corresponding logical sentence is:

$2>1$ *and we have that* $2 \not> 1$ *or* $2+2=4$.

The second set of problems deals with functional dependency and definability of propositional connectives. It extends Exercises 1 and 2 of subsection 3.3.

Exercise 3

Find an equality and a table defining implication in terms of disjunction and negation.

Solution

Definition of $(A \Rightarrow B)$ in terms of $\neg$ and $\cup$.

The equality

$$(A \Rightarrow B) = (\neg A \cup B) \tag{26}$$

defines implication in terms of negation and disjunction. We check its correctness by constructing the following table.

A	B	$(A \Rightarrow B)$	$(\neg A \cup B)$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

(27)

Exercise 4

Find an equality and a table defining conjunction in terms of disjunction and negation.

Solution

Definition of $(A \cap B)$ in terms of $\neg$ and $\cup$.

The equality

$$(A \cap B) = \neg(\neg A \cup \neg B) \tag{28}$$

defines disjunction $\cap$ in terms of $\cup$ and $\neg$. We check its correctness by constructing the following table.

A	B	$(A \cap B)$	$\neg(\neg A \cup \neg B)$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

(29)

Exercise 5

Find an equality and a table defining conjunction in terms of implication and negation.

Solution

Definition of $\cap$ in terms of $\neg$ and $\Rightarrow$.

The equality

$$(A \cap B) = \neg(A \Rightarrow \neg B) \tag{30}$$

defines conjunction in terms of implication and negation. We check its correctness by constructing the following table.

A	B	$(A \cap B)$	$\neg(A \Rightarrow \neg B)$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

(31)

Homework Problems

1. For the following sentences write their corresponding formulas.

   (1) If Mr. Smith is happy, Mrs. Smith is not happy, and if If Mr. Smith is not happy, Mrs. Smith is not happy.

   (2) If John doesn't know logic, then if he knows logic, he was born in the 12th century.

   (3) If from the fact that all sides of a triangle ABC are equal we can deduce that all angles of the triangle ABC are equal and all angles of the triangle ABC are not equal, then all sides of a triangle ABC are equal.

   (4) If it is not the fact that a line L is parallel to a line M or a line P is not parallel the line M, then the line L is not parallel to the line M or the line P is parallel the line M.

   (5) If a number a is divisible by 3 and by 5, then from the fact that it is not divisible by 3, we can deduce that it is also not divisible by 5.

2. For each of the following formulas write 3 corresponding sentences.

   (1) $(a \Rightarrow (\neg a \cap b))$.

   (2) $(((p \cup q) \cap \neg p) \Rightarrow q)$.

   (3) $((a \Rightarrow b) \Rightarrow (a \Rightarrow (b \cup c)))$.

   (4) $\neg(p \cap (\neg p \cap q))$.

   (5) $((a \Rightarrow ((\neg b \cap b) \Rightarrow c))$.

3. Prove that $\cup$ can be defined in terms of $\Rightarrow$ alone.

4. Find an equality defining $\Rightarrow$ in terms of $\uparrow$.

5. Define $\Rightarrow$ in terms of $\neg$ and $\cap$.

6. Find an equality defining $\Rightarrow$ in terms of $\downarrow$.

7. Define $\cap$ in terms of $\Rightarrow$ and $\neg$.

8. Find an equality defining $\cap$ in terms of $\downarrow$ alone.