CHAPTER 5

SOME EXTENSIONAL MANY VALUED SEMANTICS

Many valued logics in general and 3-valued logics in particular is an old object of study which had its beginning in the work of Łukasiewicz (1920). He was the first to define a 3- valued semantics for a language $\mathcal{L}_{\neg,\cap,\cup,\Rightarrow}$ of classical logic, and called it *a three valued logic* for short. He left the problem of finding a proper axiomatic proof system for it (i.e. complete with respect to his semantics) open.

The same happened to all other logics presented here. They were first defined only semantically, i.e. by providing a semantics for their languages, without a proper proof systems. We can think about the process of their creation was inverse to the creation of Classical Logic, Modal Logics, the Intuitionistic Logic which existed as axiomatic systems longtime before invention of their formal semantics.

Recently there is a revived interest in this topic, due to its potential applications in several areas in Computer Science, like: proving correctness of programs, knowledge bases, and Artificial Intelligence.

Three valued logics, when defined semantically, enlist a third logical value, besides classical T, F. We denote this third value by $\perp$. We also often assume that the third value is intermediate between truth and falsity, i.e. that $F < \perp < T$.

There has been many of proposals relating both to the intuitive interpretation of this third value $\perp$ and for the change to logic that follows in its wake. For obvious reasons all of presented here semantics take T as designated value, i.e. the value that defines the notion of satisfiability and tautology.

If T is the only designated value, the third value $\perp$ corresponds to some notion of *incomplete information*, like *undefined* or *unknown* and is often denoted by the symbol U or I.

If, on the other hand, $\perp$ corresponds to *inconsistent information*, i.e. its meaning is something like *known to be both true and false* then it takes both T and the third logical value $\perp$ as designated.

We present here only five 3-valued logics semantics that belong to the first category and are named after their authors: *Łukasiewicz, Kleene, Heyting*, and *Bochvar* logics.

Historically all these semantics were called logics, so we use the name logic for them, instead saying each time "logic defined semantically", or "semantics for a given logic".

1 Łukasiewicz Logic **Ł**

Ł stands for Łukasiewicz logic semantics. This was the first 3-valued logic semantics ever to be invented.

Motivation

Łukasiewicz developed his semantics (called logic) to deal with future contingent statements. According to him, such statements are not just neither true nor false but are indeterminate in some metaphysical sense. It is not only that we do not know their truth value but rather that they do not possess one. Intuitively, $\perp$ signifies that the statement cannot be assigned the value true of false; it is not simply that we do not have sufficient information to decide the truth value but rather the statement does not have one.

The Language

The language of **Ł** -logic is defined by the set of connectives: $\neg$ called negation, $\cup$ called disjunction, $\cap$ called conjunction and $\Rightarrow$ called implication. i.e. the language of **Ł** is

$$\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}.$$

Logical Connectives

We define the logical connectives $\neg, \Rightarrow, \cup, \cap$ of **Ł** as the following operations in the set $\{F, \perp, T\}$, where $\{F < \perp < T\}$.

$$\neg : \quad \{F, \perp, T\} \longrightarrow \{F, \perp, T\},$$

such that

$$\neg \perp = \perp, \quad \neg F = T, \quad \neg T = F.$$

$$\cap : \quad \{F, \perp, T\} \times \{F, \perp, T\} \longrightarrow \{F, \perp, T\}$$

such that for any $a, b \in \{F, \perp, T\}$,

$$a \cap b = min\{a, b\}.$$

$$\cup : \quad \{F, \perp, T\} \times \{F, \perp, T\} \longrightarrow \{F, \perp, T\}$$

such that for any $a, b \in \{F, \perp, T\}$,

$$a \cup b = max\{a, b\}.$$

$$\Rightarrow: \quad \{F, \perp, T\} \times \{F, \perp, T\} \longrightarrow \{F, \perp, T\},$$

such that for any $a, b \in \{F, \perp, T\}$,

$$a \Rightarrow b = \begin{cases} \neg a \cup b & \text{if } a > b \\ T & \text{otherwise} \end{cases}$$

It an easy exercise to verify that the above definition defines the following 3-valued truth tables.

Truth tables for Ł connectives (1)

Ł Negation

$\neg$	F	$\perp$	T
	T	$\perp$	F

Ł Disjunction

$\cup$	F	$\perp$	T
F	F	$\perp$	T
$\perp$	$\perp$	$\perp$	T
T	T	T	T

Ł Conjunction

$\cap$	F	$\perp$	T
F	F	F	F
$\perp$	F	$\perp$	$\perp$
T	F	$\perp$	T

Ł-Implication

$\Rightarrow$	F	$\perp$	T
F	T	T	T
$\perp$	$\perp$	T	T
T	F	$\perp$	T

We define, as usual, a **truth assignment** v, as a function

$$v : VAR \longrightarrow \{F, \perp, T\}$$

and denote by v^* its extension to the set $\mathcal{F}$ of all formulas. I.e.

$$v^* : \mathcal{F} \longrightarrow \{F, \perp, T\}.$$

Definition 1.1 (Ł semantics) *For any truth assignment $v : VAR \longrightarrow \{F, \perp, T\}$, we define $v^* : \mathcal{F} \longrightarrow \{F, \perp, T\}$ by the induction on the degree of formulas as follows.*

$v^*(a) = v(a)$, *for $a \in VAR$,*

$v^*(\neg A) = \neg v^*(A)$,

$v^*(A \cap B) = (v^*(A) \cap v^*(B))$,

$v^*(A \cup B) = (v^*(A) \cup v^*(B))$,

$v^*(A \Rightarrow B) = (v^*(A) \Rightarrow v^*(B))$.

We remind that $\neg$ on the left side denotes the negation connective of $\mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$ and $\neg$ on the right side of the equation denotes the operation in the set $\{F, \perp, T\}$ as defined by the negation table 1 .

Similarly, for all reminding connectives: $\Rightarrow, \cup, \cap$ on the left side of the equation denote the connectives of $\mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$ and on the right side of the equation they denote the respective two argument operations in the set $\{F, \perp, T\}$ as defined by the tables in 1 .

We define notions of satisfaction, model, counter-model, tautology in a similar way as in the case of classical semantics logic, i.e. we adopt the following definitions.

Definition 1.2 (Ł Model, Counter- Model) *Any truth assignment* v,

$v : VAR \longrightarrow \{F, \perp, T\}$ *such that* $v^*(A) = T$ *is called a* **Ł** *model for the formula* $A \in \mathcal{F}$.

Any $v : VAR \longrightarrow \{F, \perp, T\}$ *such that* $v^*(A) \neq T$ *is called a* **Ł** *counter-model for* A.

Definition 1.3 (Ł Tautologies) *For any* $A \in \mathcal{F}$,

A *is an* **Ł** *tautology iff* $v^*(A) = T$, *for all* $v : VAR \longrightarrow \{F, \perp, T\}$,

i.e. if all truth assignments v *are* **Ł** *models for* A.

To be able to distinguish, when we need, classical tautologies and **Ł** tautologies, we write

$$\models_{\text{Ł}} A$$

to denote that A is an **Ł** tautology.

We, prove, in the same way as in Chapter 4, the following theorem that justifies the truth table method of verification of Ł tautologies.

Theorem 1.1 (Ł Tautology) *For any formula* $A \in \mathcal{F}$,

$$\models_{L} A \quad if\ and\ only\ if$$

$$v \models_{L} A \ \ for\ all\ v,\ such\ that\ \ v : VAR_A \longrightarrow \{F, \perp, T\}.$$

Theorem 1.2 (Not Ł Tautology) *For any formula* $A \in \mathcal{F}$,

$$\not\models_{L} \quad if\ and\ only\ if$$

$$v \not\models_{L} \ \ for\ some\ v,\ such\ that\ \ v :\ VAR_A \longrightarrow \{F, \perp, T\}.$$

The theorems 1.1, 1.2 prove also that the notion of Ł propositional tautology is decidable, i.e. that the following holds.

Theorem 1.3 (Decidability) *For any formula $A \in \mathcal{F}$, one has examine at most*

$$3^{VAR_A}$$

truth assignments $v: \; VAR_A \;\longrightarrow\; \{F, \perp, T\}$ in order to decide whether

$$\models_{L} A, \quad \textit{or} \quad \not\models_{L} A.$$

I.e. the notion of classical propositional tautology is decidable.

Let **ŁT**, **T** denote the sets of all **Ł** and classical tautologies, respectively. I.e.

$$\text{Ł}\mathbf{T} = \{A \in \mathcal{F} : \;\; \models_{\text{Ł}} A\},$$

$$\mathbf{T} = \{A \in \mathcal{F} : \;\; \models A\}.$$

Some natural questions of the relationship of these two sets of tautologies arise:

1.Is the **Ł** logic really different from the classical logic? It means are theirs sets of tautologies different?

2. Even if we prove that they are different, maybe they do have something in common (besides the same language)? It means do they share some tautologies?

We put the answers in a form of the following theorem.

Theorem 1.4 *Let* **ŁT**, **T** *denote the set of all* **Ł** *and classical tautologies, respectively. Then the following relationship holds.*

$$Ł\mathbf{T} \neq \mathbf{T}, \tag{2}$$

$$Ł\mathbf{T} \subset \mathbf{T}. \tag{3}$$

Proof Consider a formula $(\neg a \cup a)$. It is obviously a classical tautology. Consider any variable assignment $v : VAR \longrightarrow \{F, \perp, T\}$ such that $v(a) = \perp$. By definition we have that $v^*(\neg a \cup a) = v^*(\neg a) \cup v^*(a) = \neg v(a) \cup v(a) = \neg \perp \cup \perp = \perp \cup \perp = \perp$ what proves that v is a **Ł** counter-model for $(\neg a \cup a)$. This proves the property 2.

Observe now that if we restrict the Truth Tables 1 for **Ł** connectives to the values T and F only, we get the Truth Tables for classical connectives. This means that if $v*(A) = T$ for all $v : VAR \longrightarrow \{F, \perp, T\}$, then $v*(A) = T$ for any $v : VAR \longrightarrow \{F, T\}$ and any $A \in \mathcal{F}$, i.e the property 3 also holds.

2 Kleene Logic K

Kleene's logic was originally conceived to accommodate undecided mathematical statements.

Motivation
In Kleene's semantics the third logical value $\perp$, intuitively, represents *undecided*. Its purpose is to signal a state of partial ignorance. A sentence a is assigned a value $\perp$ just in case it is not *known* to be either true of false.

For example, imagine a detective trying to solve a murder. He may conjecture that Jones killed the victim. He cannot, at present, assign a truth value T or F to his conjecture, so we assign the value $\perp$, but it is certainly either true of false and $\perp$ represents our ignorance rather then total unknown.

The Language

We adopt the same language as in a case of classical or Łukasiewicz's **Ł** logic. I.e. the language of **K** logic is:

$$\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}.$$

Logical Connectives

We assume, as in the case of **Ł** logic, that $\{F < \perp < T\}$. Moreover, the logical connectives $\neg, \cup, \cap$ of **K** are defined as in **Ł** logic, i.e. for any $a, b \in \{F, \perp, T\}$,

$$\neg \perp = \perp, \quad \neg F = T, \quad \neg T = F,$$

$$a \cup b = max\{a, b\},$$

$$a \cap b = min\{a, b\}.$$

The implication in Kleene's logic is defined as follows.

$$\Rightarrow: \quad \{F, \perp, T\} \times \{F, \perp, T\} \longrightarrow \{F, \perp, T\},$$

such that for any $a, b \in \{F, \perp, T\}$,

$$a \Rightarrow b = \neg a \cup b. \tag{4}$$

The truth tables for the **K** negation, disjunction and conjunction are the same as the corresponding tables 1.

The Kleene's 3-valued truth tables differ hence from Łukasiewicz's truth tables 1 only in a case of implication. This table is:

K-Implication

$\Rightarrow$	F	$\perp$	T
F	T	T	T
$\perp$	$\perp$	$\perp$	T
T	F	$\perp$	T

We define, as in the case of the classical Łukasiewicz semantics, a *truth assignment* v, as a function

$$v : VAR \longrightarrow \{F, \perp, T\}.$$

Definition 2.1 (K semantics) *For any truth assignment* $v : VAR \longrightarrow \{F, \perp, T\}$*, we define* $v^* : \mathcal{F} \longrightarrow \{F, \perp, T\}$ *by the induction on the degree of formulas as follows.*

$v^*(a) = v(a)$, *for* $a \in VAR$,

$v^*(\neg A) = \neg v^*(A)$,

$v^*(A \cap B) = (v^*(A) \cap v^*(B))$,

$v^*(A \cup B) = (v^*(A) \cup v^*(B))$,

$v^*(A \Rightarrow B) = (v^*(A) \Rightarrow v^*(B))$.

We remind that $\neg, \cup, \cap$ on the right side of the equation denotes the operations in the set $\{F, \perp, T\}$ as defined by the corresponding tables 1 and the $\Rightarrow$ denotes now the operation defined by the property 4 i.e. by the **K** implication table.

The notions of K model, counter-model, K tautology are defined in a similar way as for **Ł** logic.

Definition 2.2 (K Model, Counter-Model) *Any truth assignment* v, $v : VAR \longrightarrow \{F, \perp, T\}$, *such that* $v^*(A) = T$ *is called a* **K model** *for the formula* $A \in \mathcal{F}$ *and denoted*

$$v \models_{\mathbf{K}} A.$$

Any truth assignment $v : VAR \longrightarrow \{F, \perp, T\}$, *such that* $v^*(A) \neq T$ *is called a* **K counter-model** *for* A. *We denote it as*

$$v \not\models_{\mathbf{K}} A.$$

Definition 2.3 (K Tautologies) *For any formula* $A \in \mathcal{F}$, A *is a* **K tautology** *if and only if*

$$v^*(A) = T,$$

for all truth assignments $v : VAR \longrightarrow \{F, \perp, T\}$, *i.e.*

$$v \models_{\mathbf{K}} A \quad for \quad all \quad v.$$

We write

$$\models_{\mathbf{K}} A$$

to denote that A is an **K** tautology.

We prove, in the same way as in Chapter 4, the following theorem that justifies the truth table method of verification of **K** tautologies.

Theorem 2.1 (K Tautology) *For any formula* $A \in \mathcal{F}$,

$$\models_{\mathbf{K}} A \quad if \ and \ only \ if$$

$$v \models_{\mathbf{K}} A \ \ for \ all \ v, \ such \ that \ \ v : VAR_A \ \longrightarrow \ \{F, \perp, T\}.$$

Theorem 2.2 (Not K Tautology) *For any formula* $A \in \mathcal{F}$,

$$\not\models_{\mathbf{K}} \quad if \ and \ only \ if$$

$$v \not\models_{\mathbf{K}} \ \ for \ some \ v, \ such \ that \ \ v : \ VAR_A \ \longrightarrow \ \{F, \perp, T\}.$$

The theorems 2.1, 2.2 prove also that the notion of **K** propositional tautology is decidable, i.e. that the following holds.

Theorem 2.3 (Decidability) *For any formula* $A \in \mathcal{F}$, *one has examine at most*

$$3^{VAR_A}$$

truth assignments $v : \ VAR_A \ \longrightarrow \ \{F, \perp, T\}$ *in order to decide whether*

$$\models_{\mathbf{K}} A, \quad or \quad \not\models_{\mathbf{K}} A.$$

I.e. the notion of classical propositional tautology is decidable.

We write

$$\mathbf{KT} = \{A \in \mathcal{F} : \ \models_K A\}$$

to denote the set and **K** tautologies.

The following fact establishes relationship between **Ł**, **K**, and classical logic.

Theorem 2.4 *Let* **ŁT**, **T**, **K** *denote the sets of all* **Ł**, *classical, and* **K** *tautologies, respectively. Then the following relationship holds.*

$$Ł\mathbf{T} \neq \mathbf{KT}, \tag{5}$$

$$\mathbf{KT} \subset \mathbf{T}. \tag{6}$$

Proof Obviously $\models_{\text{Ł}} (a \Rightarrow a)$, but for v such that $v(a) =\perp$ we have that for **K** semantics $v^*(a \Rightarrow a) = v(a) \Rightarrow v(a) =\perp\Rightarrow\perp=\perp$. This proves that $\not\models_{\mathbf{K}} (a \Rightarrow a)$ and hence property 5 holds.

The property 6 follows directly from the the fact that, as in the **Ł** case, if we restrict the Truth Tables 1 and K- implication definition 4 for **K** connectives to the values T and F only, we get the Truth Tables for classical connectives.

3 Heyting Logic H

We call the **H** logic a Heyting logic because its connectives are defined as operations on the set $\{F, \perp, T\}$ in such a way that they form a 3-element Heyting algebra, called also a 3-element pseudo-boolean algebra.

Pseudo-boolean, or Heyting algebras provide algebraic models for the intuitionistic logic. These were the first models ever defined for the intuitionistic logic. The intuitionistic logic was defined and developed by its inventor Brouwer and his school in 1900s as a proof system only. Heyting provided first axiomatization for the intuitionistic logic.

It took another 40 years of research to provide semantics for the intuitionistic propositional logic (Heyting, McKinsey, Tarski, 1942) in a form of pseudo-boolean (Heyting) algebras. It took yet another 15 to extend it to predicate logic (Rasiowa, Mostowski, 1957).

The other type of models, called Kripke Models were defined by Kripke in 1964 and were proved later to be equivalent to the pseudo-boolen models.

We say that formula A is an intutionistic tautology if and only if it is true in *all pseudo-boolean (Heying) algebras*. Hence, if A is an intuitionistic tatology (true in all algebras) is also true in a 3-element Heyting algebra (a particular algebra). From that we get that all intuitionistic propositional logic tautologies are Heyting 3-valued logic tautologies.

If we denote by **IT**, HT the sets of all tautologies of the intuitionistic semantics and Heyting 3-valued semantics, respectively we can write it symbolically as:

$$\mathbf{IT} \subset \mathbf{HT}. \tag{7}$$

From above we conclude that for any formula A,

$$If \quad \not\models_H A \ \ then \not\models_I A.$$

It means if we can show that a formula A has a Heying 3-valued counter-model, then we proved that it is not an intuitionistic tautology.

We define the **H** logic semantically as follows.

The Language

The language of **H** is the same as in previous cases i.e.

$$\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}.$$

Logical Connectives

We define the logical connectives $\neg, \Rightarrow, \cup, \cap$ of **H** as the following operations in the set $\{F, \perp, T\}$, where $\{F < \perp < T\}$.

We write them in a form of tables, usually called the classical truth tables. The definition of the operations $\cup$ and $\cap$ is the same as in the case of **Ł** and **K** logics, i.e. for any $a, b \in \{F, \perp, T\}$ we define

$$a \cup b = max\{a, b\},$$

$$a \cap b = min\{a, b\}.$$

The definition of implication and negation for **H** logic differs **Ł** and **K** logics and we define them as follows.

$$\Rightarrow: \quad \{F, \perp, T\} \times \{F, \perp, T\} \longrightarrow \{F, \perp, T\},$$

such that for any $a, b \in \{F, \perp, T\}$,

$$a \Rightarrow b = \begin{cases} T & \text{if } a \leq b \\ b & \text{otherwise} \end{cases}$$

$$\neg : \quad \{F, \perp, T\} \longrightarrow \{F, \perp, T\},$$

such that

$$\neg a = a \Rightarrow F.$$

The truth tables for **H** disjunction and conjunction are hence the same as corresponding tables 1 and it is an easy exercise to verify that the truth tables for **H** implication and negation are as follows.

Truth tables for H Implication and Negation (8)

H-Implication

$\Rightarrow$	F	$\perp$	T
F	T	T	T
$\perp$	F	T	T
T	F	$\perp$	T

H Negation

$\neg$	F	$\perp$	T
	T	F	F

We define, as usual, *a truth assignment* v, as a function

$$v : VAR \longrightarrow \{F, \perp, T\}$$

and the H-semantics in terms of v and its extension v^* its to the set $\mathcal{F}$ of all formulas as in all previous logics.

Definition 3.1 (H semantics) *For any truth assignment* $v : VAR \longrightarrow \{F, \perp, T\}$*, we define* $v^* : \mathcal{F} \longrightarrow \{F, \perp, T\}$ *by the induction on the degree of formulas as follows.*

$v^*(a) = v(a)$, *for* $a \in VAR$,

$v^*(\neg A) = \neg v^*(A)$,

$v^*(A \cap B) = (v^*(A) \cap v^*(B))$,

$v^*(A \cup B) = (v^*(A) \cup v^*(B))$,

$v^*(A \Rightarrow B) = (v^*(A) \Rightarrow v^*(B))$.

The $\neg, \Rightarrow, \cup, \cap$ on the right side of the equation denote the respective two argument operations in the set $\{F, \perp, T\}$ as defined by the tables 8 .

We define the notion a model and a tautology in a similar way as in the previous cases.

Definition 3.2 (H Model, H Counter-Model) *Any* $v : VAR \longrightarrow \{F, \perp, T\}$ *such that* $v^*(A) = T$ *is called a* **H** *model for the formula* $A \in \mathcal{F}$*. We write it as*

$$v \models_{\mathbf{H}} A.$$

Any $v : VAR \longrightarrow \{F, \perp, T\}$ *such that* $v^*(A) \neq T$ *is called a* **H** *counter-model for* A. *We denote it as*

$$v \not\models_{\mathbf{H}} A.$$

Definition 3.3 (H Tautology) *For any formula* $A \in \mathcal{F}$,

A is an **H** *tautology if and only if* $v^*(A) = T$, *for all* $v : VAR \longrightarrow \{F, \perp, T\}$, *i.e. if all truth assignments v are* **H** *models for* A.

We write

$$\models_{\mathbf{H}} A$$

to denote that A is an **H** tautology.

We prove, in the same way as in Chapter 4, the following theorems, similar to theorems 2.1, 2.2, 1.3, 2.3. They justify the truth table method of verification of **H** tautologies.

Theorem 3.1 (H Tautology) *For any formula* $A \in \mathcal{F}$,

$$\models_{\mathbf{H}} A \quad if\ and\ only\ if$$

$$v \models_{\mathbf{H}} A \ \ for\ all\ v,\ such\ that\ \ v : VAR_A \ \longrightarrow\ \{F, \perp, T\}.$$

Theorem 3.2 (Not K Tautology) *For any formula* $A \in \mathcal{F}$,

$$\not\models_{\mathbf{H}} \quad if\ and\ only\ if$$

$$v \not\models_{\mathbf{H}} \ \ for\ some\ v,\ such\ that\ \ v :\ VAR_A \ \longrightarrow\ \{F, \perp, T\}.$$

The theorems 3.1, 3.2 prove also that the notion of **H** propositional tautology is decidable, i.e. that the following holds.

Theorem 3.3 (Decidability) *For any formula* $A \in \mathcal{F}$, *one has examine at most*

$$3^{VAR_A}$$

truth assignments $v :\ VAR_A \ \longrightarrow\ \{F, \perp, T\}$ *in order to decide whether*

$$\models_{\mathbf{H}} A, \quad or \quad \not\models_{\mathbf{H}} A.$$

I.e. the notion of classical propositional tautology is decidable.

We denote

$$\mathbf{HT} = \{A \in \mathcal{F} : \ \models_{\mathbf{H}} A\}$$

the set of all **H** tautologies.

We also have the following relationship the classical logic, **Ł**, **K** logics and **H** logic.

Theorem 3.4

Let **HT**, **T**, **ŁT**, **KT** *denote the set of all tautologies of the* **H**, *classical,* **Ł**, *and* **K** *logic, respectively. Then the following relationship holds.*

$$\mathbf{HT} \neq \mathbf{T} \neq \L\mathbf{T} \neq \mathbf{KT}, \tag{9}$$

$$\mathbf{HT} \subset \mathbf{T}. \tag{10}$$

Proof A formula $(\neg a \cup a)$ a classical tautology and not an **H** tautology.The variable assignment $v : VAR \longrightarrow \{F, \perp, T\}$ such that $v(a) = \perp$ is an **H** counter-model. A formula $(A \Rightarrow A)$ is a **H** logic tautology but is not a **K** logic tautology. The variable assignment v such that $v(a) = v(b) = \perp$ proves that $\not\models_{\mathbf{K}} (\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$ but $\models_{\mathbf{H}} (\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$.

Observe now that if we restrict the truth tables 1, 8 for **H** connectives to the values T and F only, we get the truth tables for classical connectives. This means that if $v*(A) = T$ for all $v : VAR \longrightarrow \{F, \perp, T\}$, then $v*(A) = T$ for any $v : VAR \longrightarrow \{F, T\}$ and any $A \in \mathcal{F}$, i.e the property 9 also holds.

3.1 Bochvar 3-valued logic B

Bochvar's 3-valued logic was directly inspired by considerations relating to semantic paradoxes.

Motivation

Consider a semantic paradox given by a sentence: *this sentence is false*. If it is true it must be false, if it is false it must be true. There have been many proposals relating to how one may deal with semantic paradoxes. Bohvar's proposal adopts a strategy of a change of logic. According to Bochvar, such sentences are neither true of false but rather *paradoxical* or *meaningless*. The semantics follows the principle that the third logical value, denoted now by m is in some sense "infectious"; if one one component of the formula is assigned the value m then the formula is also assigned the value m.

Bohvar also adds an one argument *assertion operator* S that asserts the logical value of T and F, i.e. $SF = F$, $ST = T$ and it asserts that meaningfulness is false, i.e $Sm = F$.

Language

The language of **B** logic differs from all previous languages in that it contains an extra one argument assertion connective S added to the usual set $\{\neg, \Rightarrow, \cup, \cap\}$ of connectives. Hence, the language is

$$\mathcal{L} = \mathcal{L}_{\{\neg, S, \Rightarrow, \cup, \cap\}}.$$

Logical Connectives

We define the logical connectives $\neg, S, \Rightarrow, \cup, \cap$ of **B** as the operations in the set $\{F, T, m\}$ given by the following set of truth tables.

Truth tables for B connectives (11)

B Negation

$\neg$	F	m	T
	T	m	F

B Disjunction

$\cup$	F	m	T
F	F	m	T
m	m	m	m
T	T	m	T

B Conjunction

$\cap$	F	m	T
F	F	m	F
m	m	m	m
T	F	m	T

B Implication

$\Rightarrow$	F	m	T
F	T	m	T
m	m	m	m
T	F	m	T

B Assertion :

S	F	m	T
	F	F	T

The *truth assignment* v is a function

$$v : VAR \longrightarrow \{F, m, T\}.$$

The B-semantics in terms of v and its extension v^* its to the set $\mathcal{F}$ of all formulas as in all previous logics.

Definition 3.4 (B semantics) *For any truth assignment $v : VAR \longrightarrow \{F, \perp, T\}$, we define $v^* : \mathcal{F} \longrightarrow \{F, \perp, T\}$ by the induction on the degree of formulas as follows.*

$v^*(a) = v(a)$, *for $a \in VAR$,*

$v^*(\neg A) = \neg v^*(A)$,

$v^*(SA) = Sv^*(A)$,

$v^*(A \cap B) = (v^*(A) \cap v^*(B))$,

$v^*(A \cup B) = (v^*(A) \cup v^*(B))$,

$v^*(A \Rightarrow B) = (v^*(A) \Rightarrow v^*(B))$.

The symbols $\neg, S, \Rightarrow, \cup, \cap$ on the right side of the equations denote the respective operations in the set $\{F, m, T\}$ as defined by the tables 11.

We define the notion a model and a tautology in a similar way as in the previous cases.

Definition 3.5 (B Model, B Counter- Model) *Any $v : VAR \longrightarrow \{F, m, T\}$ such that $v^*(A) = T$ is called a* **B model** *for the formula $A \in \mathcal{F}$. We write is as*

$$v \models_{\mathbf{B}} A.$$

Any $v : VAR \longrightarrow \{F, m, T\}$ such that $v^*(A) \neq T$ is called a **B counter-model** *for A. We write is as*

$$v \not\models_{\mathbf{B}} A.$$

Definition 3.6 (B Tautologies) *A formula $A \in \mathcal{F}$ is a* **B** *tautology if and only if $v^*(A) = T$, for all $v : VAR \longrightarrow \{F, m, T\}$, i.e. if all variable assignments v are* **B** *models for A.*

We write

$$\models_{\mathbf{B}} A$$

to denote that A is an **B** tautology.

We, prove, in the same way as for all previous logics semantics, the following theorems that justify the truth table method of verification for **B** tautologies.

Theorem 3.5 (B Tautology) *For any formula* $A \in \mathcal{F}$,

$$\models_{\mathbf{B}} A \quad if\ and\ only\ if$$
$$v \models_{\mathbf{B}} A \ \ for\ all\ v,\ such\ that\ \ v : VAR_A \longrightarrow \{F, m, T\}.$$

Theorem 3.6 (Not B Tautology) *For any formula* $A \in \mathcal{F}$,

$$\not\models_{\mathbf{B}} \quad if\ and\ only\ if$$
$$v \not\models_{\mathbf{B}} \ \ for\ some\ v,\ such\ that\ \ v :\ VAR_A \longrightarrow \{F, m, T\}.$$

As in previous cases, the theorems 3.5, 3.6 prove also that the notion of **B** propositional tautology is decidable, i.e. that the following holds.

Theorem 3.7 (Decidability) *For any formula* $A \in \mathcal{F}$, *one has examine at most*

$$3^{VAR_A}$$

truth assignments $v :\ VAR_A \longrightarrow \{F, m, T\}$ *in order to decide whether*

$$\models_{\mathbf{B}} A, \quad or \quad \not\models_{\mathbf{B}} A.$$

I.e. the notion of classical propositional tautology is decidable.

Let denote by **BT** the set of all **B** tautologies:

$$\mathbf{BT} = \{A \in \mathcal{F} :\ \ \models_{\mathbf{B}} A\}.$$

The form of the **B** tautologies is more complicated to determine then in the previous semantics.

Observe that none of the formulas of $\mathcal{L}_{\{\neg,\Rightarrow,\cup,\cap\}}$ is a **B** tautology, as any v such that $v(a) = m$ for at least one variable in a formula of $\mathcal{L}_{\{\neg,\Rightarrow,\cup,\cap\}}$ is a counter-model for that formula. I. e we have that

$$\mathbf{T} \cap \mathbf{BT} = \emptyset. \tag{12}$$

So for a formula to be a tautology, it must contain the connective S. Let's look at some examples.

$$\not\models_{\mathbf{B}} (a \cup \neg a)$$

as $v(a) = m$ gives: $m \cup \neg m = m$. For the same reason we have that

$$\not\models_{\mathbf{B}} (a \cup \neg Sa),$$

$$\not\models_{\mathbf{B}} (Sa \cup \neg a),$$

$$\not\models_{\mathbf{B}} (Sa \cup S\neg a),$$

but it is easy to verify that

$$\models_{\mathbf{B}} (Sa \cup \neg Sa).$$

4 Exercises and Homework Problems

Exercise 1 **Use the fact** that $v : VAR \longrightarrow \{F, \perp, T\}$ be such that $v^*((a \cap b) \Rightarrow (a \Rightarrow c)) = \perp$
under H semantics **to evaluate** $v^*(((b \Rightarrow a) \Rightarrow (a \Rightarrow \neg c)) \cup (a \Rightarrow b))$.

Solution: $v^*((a \cap b) \Rightarrow (a \Rightarrow c)) = \perp$ under H semantics if and only if (we use shorthand notation) $(a \cap b) = T$ and $(a \Rightarrow c) = \perp$ iff $a = T, b = T$ and $(T \Rightarrow c) = \perp$ iff $c = \perp$. I.e. we have that

$$v^*((a \cap b) \Rightarrow (a \Rightarrow c)) = \perp \quad iff \quad a = T,\ b = T, c = \perp .$$

Now we can we **evaluate** $v^*(((b \Rightarrow a) \Rightarrow (a \Rightarrow \neg c)) \cup (a \Rightarrow b))$ as follows (in shorthand notation).
$v^*(((b \Rightarrow a) \Rightarrow (a \Rightarrow \neg c)) \cup (a \Rightarrow b)) =$
$(((T \Rightarrow T) \Rightarrow (T \Rightarrow \neg \perp)) \cup (T \Rightarrow T)) =$
$((T \Rightarrow (T \Rightarrow \perp)) \cup T) = T$.

Exercise 2 **Use the fact** that $v : VAR \longrightarrow \{F, \perp, T\}$ be such that $v^*((a \cap b) \Rightarrow \neg b) = \perp$
under **Ł** semantics **to evaluate** $v^*(((b \Rightarrow \neg a) \Rightarrow (a \Rightarrow \neg b)) \cup (a \Rightarrow b))$.
Use shorthand notation.

Solution : $((a \cap b) \Rightarrow \neg b) = \perp$ in two cases.

C1 $(a \cap b) = \perp$ and $\neg b = F$.

C2 $(a \cap b) = T$ and $\neg b = \perp$.

Case C1: $\neg b = F$, i.e. $b = T$, and hence $(a \cap T) = \perp$ if and only if $a = \perp$.
We get that v is such that $v(a) = \perp$ and $v(b) = T$.

We evaluate: $v^*(((b \Rightarrow \neg a) \Rightarrow (a \Rightarrow \neg b)) \cup (a \Rightarrow b)) = (((T \Rightarrow \neg \perp) \Rightarrow (\perp \Rightarrow \neg T)) \cup (\perp \Rightarrow T)) = ((\perp \Rightarrow \perp) \cup T) = T$.

Case C2: $\neg b = \perp$, i.e. $b = \perp$, and hence $(a \cap \perp) = T$ what is impossible, hence v from case C1 is the only one.

Homework Problems

1. In all 3-valued logics semantics presented here we chose the language without the equivalence connective "$\Leftrightarrow$". Prove that in each 3-valued logics semantics case its language can be extended to a logically equivalent language containing the equivalence connective. I.e. define $\Leftrightarrow$ in terms of other connective and in each case provide a truth table for $\Leftrightarrow$, and write full definition of proper semantics.

2. Let $v : VAR \longrightarrow \{F, \perp, T\}$ be any v, such that $v^*((a \cup b) \Rightarrow (a \Rightarrow c)) = \perp$ under H semantics.

Evaluate: $v^*(((b \Rightarrow a) \Rightarrow (a \Rightarrow \neg c)) \cup (a \Rightarrow b))$.

3. For each of 3-valued logic semantics presented in this chapter, find 5 classical tautologies that are not tautologies of that logic.

4. Examine whether the classical tautologies listed below are, or are not, tautologies in each of 3-valued logic semantics defined in this chapter.

Excluded Middle

$$(A \cup \neg A)$$

De Morgan Laws:

$$\neg(A \cup B) \Rightarrow (\neg A \cap \neg B),$$

$$(\neg A \cap \neg B) \Rightarrow \neg(A \cup B)$$

$$\neg(A \cap B) \Rightarrow (\neg A \cup \neg B),$$

$$(\neg A \cup \neg B) \Rightarrow \neg(A \cap B)$$

Double Negation Laws:

$$\neg\neg A \Rightarrow A,$$

$$A \Rightarrow \neg\neg A$$

Negation of Implication Laws:

$$\neg(A \Rightarrow B) \Rightarrow (A \cap \neg B),$$

$$(A \cap \neg B) \Rightarrow \neg(A \Rightarrow B).$$

5. Examine the notion of *definability of connectives* as defined in Chapter 2 for each of 3-valued logic semantics.