CHAPTER 8

Hilbert Proof Systems, Formal Proofs, Deduction Theorem

The Hilbert proof systems are systems based on a language with implication and contain a Modus Ponens rule as a rule of inference. They are usually called Hilbert style formalizations. We will call them here Hilbert style proof systems, or Hilbert systems, for short.

Modus Ponens is probably the oldest of all known rules of inference as it was already known to the Stoics (3rd century B.C.). It is also considered as the most "natural" to our intuitive thinking and the proof systems containing it as the inference rule play a special role in logic. The Hilbert proof systems put major emphasis on logical axioms, keeping the rules of inference to minimum, often in propositional case, admitting only Modus Ponens, as the sole inference rule.

1 Hilbert System H_1

Hilbert proof system H_1 is a simple proof system based on a language with implication as the only connective, with two axioms (axiom schemas) which characterize the implication, and with Modus Ponens as a sole rule of inference.

We define H_1 as follows.

$$H_1 = (\ \mathcal{L}_{\{\Rightarrow\}},\ \ \mathcal{F}\ \ \{A1, A2\}\ \ MP\ ) \tag{1}$$

where $A1, A2$ are axioms of the system, MP is its rule of inference, called Modus Ponens, defined as follows:

A1 $\quad (A \Rightarrow (B \Rightarrow A))$,

A2 $\quad ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

MP

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B},$$

and A, B, C are any formulas of the propositional language $\mathcal{L}_{\{\Rightarrow\}}$.

Finding formal proofs in this system requires some ingenuity. Let's construct, as an example, the formal proof of such a simple formula as $A \Rightarrow A$.

Example 1

The formal proof of $(A \Rightarrow A)$ in H_1 is a sequence

$$B_1,\ B_2,\ B_3,\ B_4, B_5 \tag{2}$$

as defined below.

$B_1 =\ ((A \Rightarrow ((A \Rightarrow A) \Rightarrow A)) \Rightarrow ((A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A)))$,
axiom A2 for $A = A$, $B = (A \Rightarrow A)$, and $C = A$

$B_2 =\ (A \Rightarrow ((A \Rightarrow A) \Rightarrow A))$,
axiom A1 for $A = A$, $B = (A \Rightarrow A)$

$B_3 =\ ((A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A)))$,
MP application to B_1 and B_2

$B_4 =\ (A \Rightarrow (A \Rightarrow A))$,
axiom A1 for $A = A$, $B = A$

$B_5 =\ (A \Rightarrow A)$
MP application to B_3 and B_4

We have hence proved the following.

Lemma 1.1 *For any $A \in \mathcal{F}$,*

$$\vdash_{H_1} (A \Rightarrow A)$$

and the sequence 2 constitutes its formal proof.

It is easy to see that the above proof wasn't constructed automatically. The main step in its construction was the choice of a proper form (substitution) of logical axioms to start with, and to continue the proof with. This choice is far from obvious for un-experienced prover and impossible for a machine, as the number of possible substitutions is infinite.

Observe that the systems $S_1 - S_4$ from the previous chapter were syntactically decidable for one simple reason. Their inference rules were such that it was possible to "reverse" their use; to use them in the reverse manner in order to search for proofs, and we were able to do so in a blind, fully automatic way. We were able to conduct an argument of the type: *if this formula has a proof the only way to construct it is from such and such formulas by the means of one of the inference rules, and that formula can be found automatically.*

We will see now, that one can't apply the above argument to the proof search in Hilbert proof systems, which contain Modus Ponens as an inference rule.

A general procedure for searching for proofs in a proof system S can be stated is as follows. Given an expression B of the system S. If it has a proof, it must be conclusion of the inference rule. Let's say it is a rule r. We find its premisses, with B being the conclusion, i.e. we evaluate $r^{-1}(B)$. If all premisses are axioms, the proof is found. Otherwise we repeat the procedure for any non-axiom premiss.

Search for proof in Hilbert Systems must involve the Modus Ponens. The rule says: given two formulas A and $(A \Rightarrow B)$ we can conclude a formula B.

Assume now that we have a formula B and want to find its proof. If it is an axiom, we have the proof: the formula itself. If it is not an axiom, it had to be obtained by the application of the Modus Ponens rule, to certain two formulas A and $(A \Rightarrow B)$. But there is infinitely many of formulas A and $(A \Rightarrow B)$. I.e. for any B, the inverse image of B under the rule MP, $MP^{-1}(B)$ is countably infinite.

Obviously, we have the following.

Fact 1.1 *Any Hilbert proof system is not syntactically decidable, in particular, the system H_1 is not syntactically decidable.*

Semantic Link 1 System H_1 is obviously sound under classical semantics and is sound under **Ł**, **H** semantics and not sound under **K** semantics.

We leave the proof of the following theorem (by induction with respect of the length of the formal proof) as an easy exercise to the reader.

Theorem 1.1 (Soundness of H_1) *For any $A \in \mathcal{F}$ of H_1,*

$$If \quad \vdash_{H_1} A, \quad then \quad \models A.$$

Semantic Link 2 The system H_1 **is not complete** under classical semantics. It means that not all classical tautologies have a proof in H_1. We have proved that one needs negation and one of other connectives $\cup, \cap, \Rightarrow$ to express all classical connectives, and hence all classical tautologies. Our language contains only implication and one can't express negation in terms of implication and hence we can't provide a proof of any tautology i.e. its logically equivalent form in our language.

We have constructed a formal proof 2 of $(A \Rightarrow A)$ in H_1 on a base of logical axioms, as an example of complexity of finding proofs in Hilbert systems.

In order to make the construction of formal proofs easier by the use of previously proved formulas we use the notions of a formal proof from some hypotheses Γ (and logical axioms), as defined in chapter 7. Here is a simple example.

Example 2

Construct a proof of $(A \Rightarrow C)$ from hypotheses $\Gamma = \{(A \Rightarrow B), (B \Rightarrow C)\}$. I.e. show that

$$(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C).$$

The formal proof is a sequence

$$B_1, B_2, ...B_7 \tag{3}$$

such that

$B_1 =$ $(B \Rightarrow C)$,
hypothesis

$B_2 =$ $(A \Rightarrow B)$,
hypothesis

$B_3 =$ $((B \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C)))$,
axiom A1 for $A = (B \Rightarrow C), B = A$

$B_4 =$ $(A \Rightarrow (B \Rightarrow C))$
B_1, B_3 and MP

$B_5 =$ $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,
axiom A2

$B_6 =$ $((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$,
B_5 and B_4 and MP

$B_7 = \ (A \Rightarrow C)$
B_2 and B_6 and MP

Example 2

Show, by constructing a formal proof that

$$A \vdash_{H_1} (A \Rightarrow A)$$

The formal proof is a sequence

$$B_1, B_2, B_3 \tag{4}$$

such that

$B_1 = \ A,$
hypothesis

$B_2 = \ (A \Rightarrow (A \Rightarrow A)),$
Axiom **A1** for $B = A$,

$B_3 = \ (A \Rightarrow A)$
B_1, B_2 and MP.

We can even further simplify the task of constructing formal proofs by the use of the Deduction Theorem, which is presented and proved in the next section.

2 Deduction Theorem

In mathematical arguments, one often assumes a statement B on the assumption (hypothesis) of some other statement A and then concludes that we have proved the implication "if A, then B". This reasoning is justified by the following theorem, called a Deduction Theorem. It was first formulated and proved for a proof system for the classical propositional logic by Herbrand in 1930.

Theorem 2.1 (Herbrand,1930) *For any formulas A, B,*

$$if \quad A \vdash B, \quad then \quad \vdash (A \Rightarrow B).$$

We are going to prove now that for our system H_1 is strong enough to prove the Deduction Theorem for it. In fact we prove a more general version of Herbrand theorem. To formulate it we introduce the following notation. We write $\Gamma, A \vdash B$ for $\Gamma \cup \{A\} \vdash B$, and in general we write $\Gamma, A_1, A_2, ..., A_n \vdash B$ for $\Gamma \cup \{A_1, A_2, ..., A_n\} \vdash B$.

Theorem 2.2 (Deduction Theorem for H_1) *For any subset Γ of the set of formulas $\mathcal{F}$ of H_1 and for any formulas $A, B \in \mathcal{F}$,*

$$\Gamma,\ A \vdash_{H_1} B \ \ if\ and\ only\ if\ \ \Gamma \vdash_{H_1} (A \Rightarrow B).$$

In particular,

$$A \vdash_{H_1} B \ \ if\ and\ only\ if\ \ \vdash_{H_1} (A \Rightarrow B).$$

Proof. We use in the proof the symbol $\vdash$ instead of $\vdash_{H_1}$.

Assume that $\Gamma,\ A\ \vdash B$, i.e. that we have a formal proof

$$B_1, B_2, ..., B_n \tag{5}$$

of B from the set of formulas $\Gamma \cup \{A\}$. In order to prove that $\Gamma \vdash (A \Rightarrow B)$ we will prove a little bit stronger statement, namely that $\Gamma \vdash (A \Rightarrow B_i)$ for any B_i $(1 \leq i \leq n)$ in the formal proof 5 of B. And hence, in particular case, when $i = n$, we will obtain that also $\Gamma \vdash (A \Rightarrow B)$.

The proof is conducted by induction on i ($1 \leq i \leq n$).

Step $i = 1$. When $i = 1$, it means that the formal proof 5 contains only one element B_1. By the definition of the formal proof from $\Gamma \cup \{A\}$, we have that B_1 must be an logical axiom, or in in Γ, or $B_1 = A$, i.e. $B_1 \in \{A1, A2\} \cup \Gamma \cup \{A\}$. Here we have two cases.

Case 1: $B_1 \in \{A1, A2\} \cup \Gamma$. Observe that $(B_1 \Rightarrow (A \Rightarrow B_1))$ is the axiom $A1$ and by assumption $B_1 \in \{A1, A2\} \cup \Gamma$, hence we get the required proof of $(A \Rightarrow B_1)$ from Γ by the following application of the Modus Ponens rule

$$(MP)\ \frac{B_1\ ;\ (B_1 \Rightarrow (A \Rightarrow B_1))}{(A \Rightarrow B_1)}.$$

Case 2: $B_1 = A$. When $B_1 = A$, then to prove $\Gamma \vdash (A \Rightarrow B)$ means to prove $\Gamma \vdash (A \Rightarrow A)$, what holds by the monotonicity of the consequence and the fact that we have shown that $\vdash (A \Rightarrow A)$.

The above cases conclude the proof of $\Gamma \vdash (A \Rightarrow B_i)$ for $i = 1$.

Inductive step. Assume that $\Gamma \vdash (A \Rightarrow B_k)$ for all $k < i$, we will show that using this fact we can conclude that also $\Gamma \vdash (A \Rightarrow B_i)$.

Consider a formula B_i in the sequence 5. By the definition, $B_i \in \{A1, A2\} \cup \Gamma \cup \{A\}$ or B_i follows by MP from certain B_j, B_m such that $j < m < i$. We have to consider again two cases.

Case 1: $B_i \in \{A1, A2\} \cup \Gamma \cup \{A\}$**.** The proof of $(A \Rightarrow B_i)$ from Γ in this case is obtained from the proof of the **Step** $i = 1$ by replacement B_1 by B_i and will be omitted here as a straightforward repetition.

Case 2: B_i **is a conclusion of MP.** If B_i is a conclusion of MP, then we must have two formulas B_j, B_m in the sequence 5 such that $j < m < i$ and $(MP)\ \frac{B_j\ ;\ B_m}{B_i}$. By the inductive assumption, the formulas B_j, B_m are such that

$$\Gamma \vdash (A \Rightarrow B_j) \tag{6}$$

and

$$\Gamma \vdash (A \Rightarrow B_m). \tag{7}$$

Moreover, by the definition of the Modus Ponens rule, the formula B_m has to have a form $(B_j \Rightarrow B_i)$, i.e. $B_m = (B_j \Rightarrow B_i)$, and the inductive assumption 7 can be re-written as follows.

$$\Gamma \vdash (A \Rightarrow (B_j \Rightarrow B_i)), \quad for\ j < i. \tag{8}$$

Observe now that the formula $((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))$ is a substitution of the axiom schema A2 and hence has a proof in our system. By the monotonicity of the consequence, it also has a proof from the set Γ, i.e.

$$\Gamma \vdash ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))). \tag{9}$$

Applying the rule MP to formulas 9 and 8, i.e. performing the following

$$(MP)\ \frac{(A \Rightarrow (B_j \Rightarrow B_i))\ ;\ ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))}{((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))}$$

we get that also

$$\Gamma \vdash ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)). \tag{10}$$

Applying again the rule MP to formulas 6 and 10, i.e. performing the following

$$(MP)\ \frac{(A \Rightarrow B_j)\ ;\ ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))}{(A \Rightarrow B_i)}$$

we get that

$$\Gamma \vdash (A \Rightarrow B_i)$$

what ends the proof of the inductive step. By the mathematical induction principle, we hence have proved that $\Gamma \vdash (A \Rightarrow B_j)$ for all i such that $1 \leq i \leq n$. In particular it is true for $i = n$, what means for $B_n = B$. This ends the proof of the fact that if $\Gamma, A \vdash B$, then $\Gamma \vdash (A \Rightarrow B)$.

The proof of the inverse implication is straightforward. Assume that $\Gamma \vdash (A \Rightarrow B)$, hence by the monotonicity of the consequence we have also that $\Gamma, A \vdash (A \Rightarrow B)$. Obviously, $\Gamma, A \vdash A$. Applying Modus Ponens to the above, we get the proof of B from $\{\Gamma, A\}$ i.e. we have proved that $\Gamma, A \vdash B$. That ends the proof of the deduction theorem for any set $\Gamma \subseteq \mathcal{F}$ and any formulas $A, B \in \mathcal{F}$. The particular case is obtained from the above by assuming that the set Γ is empty.

The proof of the following Lemma provides a good example of multiple applications of Deduction Theorem.

Lemma 2.1 *For any* $A, B, C \in \mathcal{F}$,

(a) $(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C)$,

(b) $(A \Rightarrow (B \Rightarrow C)) \vdash_{H_1} (B \Rightarrow (A \Rightarrow C))$.

Proof of **(a)**.

Deduction theorem says:

$(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C)$ *if and only if* $(A \Rightarrow B), (B \Rightarrow C), A \vdash_{H_1} C$.

We construct a formal proof

$$B_1, B_2, B_3, B_4, B_5$$

of $(A \Rightarrow B), (B \Rightarrow C), A \vdash_{H_1} C$ as follows.

$B_1 = (A \Rightarrow B)$,
hypothesis

$B_2 = (B \Rightarrow C)$,
hypothesis

$B_3 = A$
hypothesis

$B_4 = B$
B_1, B_3 and MP

$B_5 = C$
B_2, B_4 and MP

Thus

$$(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C)$$

by Deduction Theorem.

Proof of **(b)**.

By Deduction Theorem,

$(A \Rightarrow (B \Rightarrow C)) \vdash_{H_1} (B \Rightarrow (A \Rightarrow C))\ if\ and\ only\ if\ (A \Rightarrow (B \Rightarrow C)), B \vdash_{H_1} (A \Rightarrow C)$.

We construct a formal proof

$$B_1, B_2, B_3, B_4, B_5, B_6, B_7$$

of $(A \Rightarrow (B \Rightarrow C)), B \vdash_{H_1} (A \Rightarrow C)$. as follows.

$B_1 = (A \Rightarrow (B \Rightarrow C))$
hypothesis

$B_2 = B$
hypothesis

$B_3 = ((B \Rightarrow (A \Rightarrow B))$
$A1$ for $A = B, B = A$

$B_4 = (A \Rightarrow B)$
B_2, B_3 and MP

$B_5 = ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$
$axiom A2$

$B_6 = ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$
B_1, B_5 and MP

$B_7 = \ (A \Rightarrow C)$

Thus

$$(A \Rightarrow (B \Rightarrow C)) \vdash_{H_1} (B \Rightarrow (A \Rightarrow C))$$

by Deduction Theorem.

3 Hilbert System H_2

The system H_1 presented in the previous section is sound and strong enough to prove the Deduction Theorem for it, but it is not complete.

We extend now its set of logical axioms to *a complete set of axioms*, i.e. we define a system H_2 that is *complete* with respect to classical semantics. The proof of completeness will be presented in the next chapter.

H_2 is the following proof system:

$$H_2 = (\ \mathcal{L}_{\{\Rightarrow,\neg\}}, \quad A1, A2, A3, \quad MP\ ) \tag{11}$$

where $A1, A2, A3$ are axioms of the system defined below, MP is its rule of inference, called Modus Ponens is called a Hilbert proof system for the classical propositional logic. The axioms $A1 - A3$ are defined as follows.

A1 $(A \Rightarrow (B \Rightarrow A))$,

A2 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$

MP (Rule of inference)

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B},$$

and A, B, C are any formulas of the propositional language $\mathcal{L}_{\{\Rightarrow,\neg\}}$.

We write, as before

$$\vdash_{H_2} A$$

to denote that a formula A has a formal proof in H_2 (from the set of logical axioms $A1, A2, A3$). We write

$$\Gamma \vdash_{H_2} A$$

to denote that a formula A has a formal proof in H_2 from a set of formulas Γ (and the set of logical axioms $A1, A2, A3$).

Observe that system H_2 was obtained by adding axiom A_3 to the system H_1. Hence the Deduction Theorem holds for system H_2.

Theorem 3.1 (Deduction Theorem for H_2) *For any subset Γ of the set of formulas $\mathcal{F}$ of H_2 and for any formulas $A, B \in \mathcal{F}$,*

$$\Gamma,\ A \vdash_{H_2} B \quad if\ and\ only\ if \quad \Gamma \vdash_{H_2} (A \Rightarrow B).$$

In particular,

$$A \vdash_{H_2} B \quad if\ and\ only\ if \quad \vdash_{H_2} (A \Rightarrow B).$$

Obviously, the axioms $A1, A2, A3$ are tautologies, and the Modus Ponens rule leads from tautologies to tautologies, hence our proof system H_2 is *sound* i.e. the following holds.

Theorem 3.2 (Soundness Theorem for H_2) *For every formula $A \in \mathcal{F}$,*

$$if \quad \vdash_{H_2} A, \quad then \quad \models A.$$

The soundness theorem proves that the system "produces" only tautologies. We show, in the next chapter, that our proof system H_2 "produces" not only tautologies, but that all tautologies are provable in it. This is called a *completeness theorem for classical logic.*

Theorem 3.3 (Completeness Theorem for H_2) *For every $A \in \mathcal{F}$,*

$$\vdash_{H_2} A, \quad if\ and\ only\ if \quad \models A.$$

The proof of completeness theorem (for a given semantics) is always a main point in any logic creation. There are many ways (techniques) to prove it, depending on the proof system, and on the semantics we define for it.

We present in the next chapter two proofs of the completeness theorem for our system H_2. The proofs use very different techniques, hence the reason of presenting both of them. In fact the proofs are valid for any proof system for classical propositional logic in which one can prove all formulas proved in the next section and stated in lemma 4.1.

4 Formal Proofs in H_2

We present here some examples of formal proofs in H_2. There are two reasons for presenting them. First reason is that all formulas we prove here to be provable play a crucial role in the proof of Completeness Theorem for H_2, or are needed to find formal proofs of those needed. The second reason is that they provide a "training" ground for a reader to learn how to develop formal proofs. For this second reason we write some proofs in a full detail and we leave some others for the reader to complete in a way explained in the following example.

We write $\vdash$ instead of $\vdash_{H_2}$ for the sake of simplicity.

Example 1

Here are consecutive steps

$$B_1, ..., B_5, B_6 \tag{12}$$

of the proof (in H_2) of $(\neg\neg B \Rightarrow B)$.

$B_1 =\ ((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$

$B_2 =\ ((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$

$B_3 =\ (\neg B \Rightarrow \neg B)$

$B_4 =\ ((\neg B \Rightarrow \neg\neg B) \Rightarrow B)$

$B_5 =\ (\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B))$

$B_6 =\ (\neg\neg B \Rightarrow B)$

Exercise 1

Complete the proof 12 by providing comments how each step of the proof was obtained.

Solution

The comments that complete the proof are as follows.

$B_1 =\ ((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$
Axiom $A3$ for $A = \neg B, B = B$

$B_2 =$ $((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$

B_1 and lemma 2.1 **b** for $A = (\neg B \Rightarrow \neg\neg B), B = (\neg B \Rightarrow \neg B), C = B$, i.e.

$((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B)) \vdash ((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$

$B_3 =$ $(\neg B \Rightarrow \neg B)$

Lemma 1.1 for $A = \neg B$

$B_4 =$ $((\neg B \Rightarrow \neg\neg B) \Rightarrow B)$

B_2, B_3 and MP

$B_5 =$ $(\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B))$

Axiom $A1$ for $A = \neg\neg B, B = \neg B$

$B_6 =$ $(\neg\neg B \Rightarrow B)$

B_4, B_5 and Lemma 2.1 **a** for $A = \neg\neg B, B = (\neg B \Rightarrow \neg\neg B), C = B$; i.e.

$(\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B)),\ ((\neg B \Rightarrow \neg\neg B) \Rightarrow B) \vdash (\neg\neg B \Rightarrow B)$.

General Remark

In step B_2, B_3, B_5, B_6 we call previously proved facts and use their results as a part of our proof. We can insert previously constructed formal proofs into our formal proof. For example we adopt previously constructed proof 2 of $(A \Rightarrow A)$ in H_1 to the proof of $(\neg B \Rightarrow \neg B)$ in H_2 by replacing A by $\neg B$ and we insert the proof of $(\neg B \Rightarrow \neg B)$ after B_2. The "old" step B_3 becomes now B_7, the "old" step B_4 becomes now B_8, etc.....

$B_1 =$ $((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$

Axiom $A3$ for $A = \neg B, B = B$

$B_2 =$ $((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$

B_1 and lemma 2.1 **b** for $A = (\neg B \Rightarrow \neg\neg B), B = (\neg B \Rightarrow \neg B), C = B$,

$B_3 =$ $((\neg B \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow \neg B)) \Rightarrow ((\neg B \Rightarrow (\neg B \Rightarrow \neg B)) \Rightarrow (\neg B \Rightarrow \neg B)))$,

axiom A2 for $A = \neg B$, $B = (\neg B \Rightarrow \neg B)$, and $C = \neg B$

$B_4 =$ $(\neg B \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow \neg B))$,

axiom A1 for $A = \neg B$, $B = (\neg B \Rightarrow \neg B)$

$B_5 =$ $((\neg B \Rightarrow (\neg B \Rightarrow \neg B)) \Rightarrow (\neg B \Rightarrow \neg B)))$,

MP application to B_4 and B_3

$B_6 =$ $(\neg B \Rightarrow (\neg B \Rightarrow \neg B))$,
axiom A1 for $A = \neg B$, $B = \neg B$

$B_7 =$ ("old" B_3)$(\neg B \Rightarrow \neg B)$
MP application to B_5 and B_4

$B_8 =$ ("old" B_4) $((\neg B \Rightarrow \neg\neg B) \Rightarrow B)$
B_2, B_3 and MP

$B_9 =$ ("old B_5) $(\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B))$
Axiom $A1$ for $A = \neg\neg B, B = \neg B$

$B_{10} =$ ("old B_6) $(\neg\neg B \Rightarrow B)$
B_8, B_9 and Lemma 2.1 **a** for $A = \neg\neg B, B = (\neg B \Rightarrow \neg\neg B), C = B$

We repeat our procedure by replacing the step B_2 by its formal proof as defined in the proof of the lemma 2.1 **b**, and continue the process for all other steps which involved application of lemma 2.1 until we get a full formal proof from the axioms of H_2 only.

Usually we don't need to do it, but it is important to remember that it always can be done, if we wished to take time and space to do so.

Example 2

Here are consecutive steps

$$B_1, ..., B_5 \tag{13}$$

in a proof of

$$(B \Rightarrow \neg\neg B).$$

$B_1 =$ $((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))$

$B_2 =$ $(\neg\neg\neg B \Rightarrow \neg B)$

$B_3 =$ $((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)$

$B_4 =$ $(B \Rightarrow (\neg\neg\neg B \Rightarrow B))$

$B_5 =$ $(B \Rightarrow \neg\neg B)$

Exercise 2

Complete the proof sequence 13 by providing comments how each step of the proof was obtained.

Solution

The comments that complete the proof 13 are as follows.

$B_1 = ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))$
Axiom $A3$ for $A = B, B = \neg\neg B$

$B_2 = (\neg\neg\neg B \Rightarrow \neg B)$
Example 1 for $B = \neg B$

$B_3 = ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)$
B_1, B_2 and MP, i.e.

$$\frac{(\neg\neg\neg B \Rightarrow \neg B); ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))}{((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)}$$

$B_4 = (B \Rightarrow (\neg\neg\neg B \Rightarrow B))$
Axiom $A1$ for $A = B, B = \neg\neg\neg B$

$B_5 = (B \Rightarrow \neg\neg B)$
B_3, B_4 and lemma 2.1**a** for $A = B, B = (\neg\neg\neg B \Rightarrow B), C = \neg\neg B$, i.e.

$$(B \Rightarrow (\neg\neg\neg B \Rightarrow B)), ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B) \vdash_{H_2} (B \Rightarrow \neg\neg B)$$

Example 3

Here are consecutive steps

$$B_1, ..., B_{12} \tag{14}$$

in a proof of

$$(\neg A \Rightarrow (A \Rightarrow B)).$$

$B_1 = \neg A$

$B_2 = A$

$B_3 = (A \Rightarrow (\neg B \Rightarrow A))$

$B_4 = (\neg A \Rightarrow (\neg B \Rightarrow \neg A))$

$B_5 = (\neg B \Rightarrow A)$

$B_6 = (\neg B \Rightarrow \neg A)$

$B_7 = ((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$

$B_8 = ((\neg B \Rightarrow A) \Rightarrow B)$

$B_9 = B$

$B_{10} = \neg A, A \vdash B$

$B_{11} = \neg A \vdash (A \Rightarrow B)$

$B_{12} = (\neg A \Rightarrow (A \Rightarrow B))$

Exercise 3

(1) Complete the proof sequence 14 by providing comments how each step of the proof was obtained.

(2) Prove that $\neg A, A \vdash B$.

Example 4

Here are consecutive steps

$$B_1, ..., B_7 \tag{15}$$

in a proof of

$$((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)).$$

$B_1 = (\neg B \Rightarrow \neg A)$

$B_2 = ((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$

$B_3 = (A \Rightarrow (\neg B \Rightarrow A))$

$B_4 = ((\neg B \Rightarrow A) \Rightarrow B)$

$B_5 = (A \Rightarrow B)$

$B_6 = (\neg B \Rightarrow \neg A) \vdash (A \Rightarrow B)$

$B_6 = ((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B))$

Exercise 4

Complete the proof sequence 15 by providing comments how each step of the proof was obtained.

Example 5

Here are consecutive steps

$$B_1, ..., B_9 \tag{16}$$

in a proof of

$$((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)).$$

$B_1 =\ (A \Rightarrow B)$

$B_2 =\ (\neg\neg A \Rightarrow A)$

$B_3 =\ (\neg\neg A \Rightarrow B)$

$B_4 =\ (B \Rightarrow \neg\neg B)$

$B_5 =\ (\neg\neg A \Rightarrow \neg\neg B)$

$B_6 =\ ((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A))$

$B_7 =\ (\neg B \Rightarrow \neg A)$

$B_8 =\ (A \Rightarrow B) \vdash (\neg B \Rightarrow \neg A)$

$B_9 =\ ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

Exercise 5

Complete the proof sequence 16 by providing comments how each step of the proof was obtained.

Solution

$B_1 =\ (A \Rightarrow B)$
$\quad$ *Hypothesis*

$B_2 =\ (\neg\neg A \Rightarrow A)$
$\quad$ Example 1 for $B = A$

$B_3 =\ (\neg\neg A \Rightarrow B)$
$\quad$ Lemma 2.1 **a** for $A = \neg\neg A, B = A, C = B$

$B_4 =\ (B \Rightarrow \neg\neg B)$
$\quad$ Example 2

$B_5 =\ (\neg\neg A \Rightarrow \neg\neg B)$
$\quad$ Lemma 2.1 **a** for $A = \neg\neg A, B = B, C = \neg\neg B$

$B_6 =\ ((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A))$
$\quad$ Example 4 for $B = \neg A, A = \neg B$

$B_7 =$ $(\neg B \Rightarrow \neg A)$
B_5, B_6 and MP

$B_8 =$ $(A \Rightarrow B) \vdash (\neg B \Rightarrow \neg A)$
$B_1 - B_7$

$B_9 =$ $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$
Deduction Theorem

Exercise 6

Prove that

$$\vdash (A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B)))).$$

Solution Here are consecutive steps of building the formal proof.

$B_1 =$ $A, (A \Rightarrow B) \vdash B$
by MP

$B_2 =$ $A \vdash ((A \Rightarrow B) \Rightarrow B)$
Deduction Theorem

$B_3 =$ $\vdash (A \Rightarrow ((A \Rightarrow B) \Rightarrow B))$
Deduction Theorem

$B_4 =$ $\vdash (((A \Rightarrow B) \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B)))$
Example 5 for $A = (A \Rightarrow B), B = B$

$B_5 =$ $\vdash (A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B)))$
3. and 4. and lemma **2a** for $A = A, B = ((A \Rightarrow B) \Rightarrow B), C = (\neg B \Rightarrow (\neg(A \Rightarrow B))$

Example 7

Here are consecutive steps

$$B_1, ..., B_{12} \tag{17}$$

in a proof of

$$((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B)).$$

$B_1 =$ $(A \Rightarrow B)$

$B_2 =$ $(\neg A \Rightarrow B)$

$B_3 =$ $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

$B_4 =$ $(\neg B \Rightarrow \neg A)$

$B_5 = ((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg\neg A))$

$B_6 = (\neg B \Rightarrow \neg\neg A)$

$B_7 = ((\neg B \Rightarrow \neg\neg A) \Rightarrow ((\neg B \Rightarrow \neg A) \Rightarrow B)))$

$B_8 = ((\neg B \Rightarrow \neg A) \Rightarrow B)$

$B_9 = B$

$B_{10} = (A \Rightarrow B), (\neg A \Rightarrow B) \vdash B$

$B_{11} = (A \Rightarrow B) \vdash ((\neg A \Rightarrow B) \Rightarrow B)$

$B_{12} = ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$

Exercise 7

Complete the proof sequence 17 by providing comments how each step of the proof was obtained.

Solution

$B_1 = (A \Rightarrow B)$
Hypothesis

$B_2 = (\neg A \Rightarrow B)$
Hypothesis

$B_3 = ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$
Example 5

$B_4 = (\neg B \Rightarrow \neg A)$
B_1, B_3 and MP

$B_5 = ((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg\neg A))$
Example 5 for $A = \neg A, B = B$

$B_6 = (\neg B \Rightarrow \neg\neg A)$
B_2, B_5 and MP

$B_7 = ((\neg B \Rightarrow \neg\neg A) \Rightarrow ((\neg B \Rightarrow \neg A) \Rightarrow B)))$
Axiom A3 for $B = B, A = \neg A$

$B_8 = ((\neg B \Rightarrow \neg A) \Rightarrow B)$
B_6, B_7 and MP

$B_9 = B$
B_4, B_8 and MP

$B_{10} = \ (A \Rightarrow B), (\neg A \Rightarrow B) \vdash B$
$B_1 - B_9$

$B_{11} = \ (A \Rightarrow B) \vdash ((\neg A \Rightarrow B) \Rightarrow B)$
Deduction Theorem

$B_{12} = \ ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$ Deduction Theorem

Example 8

Here are consecutive steps

$$B_1, ..., B_3 \tag{18}$$

in a proof of

$$((\neg A \Rightarrow A) \Rightarrow A).$$

$B_1 = \ ((\neg A \Rightarrow \neg A) \Rightarrow ((\neg A \Rightarrow A) \Rightarrow A)))$

$B_2 = \ (\neg A \Rightarrow \neg A)$

$B_3 = \ ((\neg A \Rightarrow A) \Rightarrow A))$

Exercise 8

Complete the proof sequence 18 by providing comments how each step of the proof was obtained.

Solution

$B_1 = \ ((\neg A \Rightarrow \neg A) \Rightarrow ((\neg A \Rightarrow A) \Rightarrow A)))$
Axiom A3 for $B = A$

$B_2 = \ (\neg A \Rightarrow \neg A)$
Lemma 1.1 for $A = \neg A$

$B_3 = \ ((\neg A \Rightarrow A) \Rightarrow A))$
B_1, B_2 and MP

The above examples 1 - 8, and the example 1 of previous section provide a proof of the following lemma.

Lemma 4.1 *For any formulas A, B, C of the system H_2,*

1. $\vdash_{H_2} (A \Rightarrow A)$

2. $\vdash_{H_2} (\neg\neg B \Rightarrow B)$

3. $\vdash_{H_2} (B \Rightarrow \neg\neg B)$

4. $\vdash_{H_2} (\neg A \Rightarrow (A \Rightarrow B))$

5. $\vdash_{H_2} ((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B))$

6. $\vdash_{H_2} ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

7. $\vdash_{H_2} (A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B)))$

8. $\vdash_{H_2} ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$

9. $\vdash_{H_2} ((\neg A \Rightarrow A) \Rightarrow A$

The set of provable formulas from the above lemma 4.1 includes a set of provable formulas (formulas 1, 3, 4, and 7-9) needed, with H_2 axioms to execute two proofs of the Completeness Theorem for H_2 which we present in the next chapter. These two proofs represent two diametrally different methods of proving Completeness Theorem.