CSE541 EXERCISE 01

EQUIVALENCE RELATIONS

QUESTION 1 Given a set $A \neq \emptyset$ and two relations defined in A, i.e. $R_1, R_2 \subseteq A \times A$.

Determine whether the following relations are, or are not equivalence relations: $R_1 \cap R_2, R_1 \cup R_2, \neg R_1 =\subseteq A \times A - R_1$.

In case when a given relation is an equivalence find its equivalence classes.

QUESTION 2 Given an indexed family of EQUIVALENCE Relations $\{R_t\}_{t \in T}$ defined in a set $A \neq \emptyset$. Determine whether

1. $\bigcap_{t \in T} R_t$ is an equivalence relation,

2. $\bigcup_{t \in T} R_t$ is an equivalence relation, and if it is not, give a counter-example.

QUESTION 3] Given sets $X, Y \neq \emptyset$ and a function $f : X \longrightarrow Y$. We define a relation $\sim_f$ on X as follows: for any $x, y \in X$

$$x \sim_f y \quad iff \quad f(x) = f(y).$$

Prove that $\sim_f$ is an equivalence. Describe the equivalence classes. Formulate the conditions for $\sim_f$ to be identity.

QUESTION 3

1. Prove the following

Theorem 1 For any $A \neq \emptyset$, and any equivalence relation $\approx$ on A, the family $A/\approx$ of sets is a partition of A, i.e.

(i) $\forall\ [a] \in A/\approx \ \ ([a] \neq \emptyset)$
i.e. all equivalence classes are non-empty.

(ii) $\forall\ [a] \neq [b] \in A/\approx \ \ ([a] \cap [b] = \emptyset)$
i.e. all equivalence classes are disjoint.

(iii) $\bigcup A/\approx = A$
i.e sum of all equivalence classes (sets from $A/\approx$) is the set A.

2. Prove the following "inverse" theorem to the Theorem 1.

Theorem 2 For any $A \neq \emptyset$ and any partition $\mathbf{P} \subseteq \mathcal{P}(A)$ of A, there is an equivalence relation on A such that its equivalence classes are exactly the sets of the partition $\mathbf{P}$.

3.

Sets R(a) Observe that we can consider, for ANY relation R on A sets that "look" like equivalence classes i.e. are defined as follows:

$R(a) = \{b \in A; \ \ aRb\} = \{b \in A; \ \ (a, b) \in R\}$.

Fact 1 R is an equivalence on A iff the family $\{R(a)\}_{a \in A}$ is a partition of A.

QUESTION 4 Given a family $\mathcal{F}$ of the following intervals of real numbers R,

$$\mathcal{F} = \{[a, a+1) : a \in Z\}.$$

Define an equivalence relation $\sim$ on R such that its equivalence classes are exactly the sets of $\mathcal{F}$. Prove that such equivalence exists.

CONSTRUCTION OF INTEGERS and RATIONAL NUMBERS

QUESTION 5 Consider the following relation $\approx$ defined on the set $N \times N$, where N is the set on natural numbers.

$$(m_1, n_1) \approx (m_2, n_2) \;\; iff \quad m_1 + n_2 = m_2 + n_1.$$

1. Prove that it is an equivalence and find equivalence classes.

2. Describe how the equivalence classes define positive and negative integers.

3 We have the following definitions of operations of multiplication and addition on those numbers:

$$[(m_1, n_1)] + [(m_2, n_2)] \;=\; [(m_1 + m_2, n_1 + n_2)]$$

$$[(m_1, n_1)] \cdot [(m_2, n_2] \;=\; [(m_1 m_2 + n_1 + n_2, m_1 n_2 + n_1 m_2)]$$

Show that they comply with all basic laws in the arithmetic of natural numbers; moreover, that the substraction can always be defined in the domain of such defined integers.

QUESTION 6 Consider the following relation $\approx$ defined on the set $Z \times Z - \{0\}$, where N is the set on natural numbers.

$$(m_1, n_1) \approx (m_2, n_2) \;\; iff \quad m_1 n_2 = m_2 n_1.$$

1. Prove that it is an equivalence and find equivalence classes.

2. Describe how the equivalence classes define rational numbers.

3 We have the following definitions of operations of multiplication and addition on those numbers:

$$[(m_1, n_1)] + [(m_2, n_2)] \;=\; [(m_1 n_2 + n_1 m_2, n_1 n_2)]$$

$$[(m_1, n_1)] \cdot [(m_2, n_2)] \;=\; [(m_1 m_2 m n_1 n_2)]$$

Show that they comply with all basic laws in the arithmetic of natural numbers; moreover, that the division by a rational number other then 0, i.e., other then $[(m, n)]$, where $m = 0$, can always be defined in the domain of such defined rational numbers.

NOTE on Cantor's Theory of Real Numbers

Let X be the set of **all sequences with rational terms** satisfying Cauchy's condition of **convergence**. Thus, a sequence $\{a_n\}_{n \in N}$ is in X iff the following condition is satisfied:

for every rational number $\varepsilon > 0$ there is natural number n_0 such that for every natural number n and for every natural number k the condition $n > n_0$ implies $|a_n - a_{n+k}| < \varepsilon$.
Let $\sim$ be an equivalence relation on X defined as:

$$\{a_n\}_{n\in N} \sim \{b_n\}_{n\in N} \quad iff \quad lim_{n\to\infty}(a_n - b_n) = 0.$$

The real numbers are defined its the equivalence classes .

POSETS and LATTICES

QUESTION 7 Prove the following

Theorem 1 In any poset $(A, \preceq)$, if a greatest and a least elements exist, then they are unique.

QUESTION 8 Prove the following

Theorem 2 If $(A, \preceq)$ is a finite poset (i.e. A is a finite set), then a unique maximal (if exists) is the largest element and a unique minimal (if exists) is the least element. item[Theorem 3] In any poset, the largest element is a unique maximal element and the least element is the unique minimal element.

QUESTION 9 Show that it is possible to to order an infinite set A in such a way that the poset $(A, \preceq)$ has a unique maximal element (minimal element) and no largest element (least element).

QUESTION 10 Show examples of a set $B \subseteq A$ of a poset $(A, \preceq)$ such that it can have none, finite or infinite number of lower or upper bounds, depending of ordering.

QUESTION 11 Prove the following theorem:

Theorem 4 If $(A, \cup, \cap)$ is a finite lattice (i.e. A is a finite set), then 1 and 0 always exist.

QUESTION 12 Show that if $(A, \cup, \cap)$ is an infinite lattice (i.e. the set A is infinite), then 1 or 0 might or might not exist.