Problem 1

  H is the following proof system:

$$H = (\ \mathcal{L}_{\{\Rightarrow,\neg\}},\ \mathcal{F},\ AX = \{A1, A2, A3\},\ \ MP\ )$$

A1 $(A \Rightarrow (B \Rightarrow A))$,

A2 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$

A4 $(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)$

MP (Rule of inference)

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

(1) Prove that H is SOUND under classical semantics.

(2) Does Deduction Theorem holds for H? Justify shortly your answer.

(3) Is H COMPLETE with respect to all classical semantics tautologies?

Problem 2 S is the following (sound) proof system:

$$S = (\ \mathcal{L}_{\{\Rightarrow,\cap\}},\ \mathcal{F},\ AX = \{A1\}\ \mathcal{R} = \{(r_1), (r_2)\}\ ),$$

   where

Axiom: $A1 = (B \Rightarrow (A \Rightarrow B))$,

Rules:

$$(r_1)\frac{A\ ;\ B}{(A \cap B)} \qquad\qquad (r_2)\frac{A\ ;\ (C \cap D)}{(A \Rightarrow (C \cap D))}$$

  For the sequence B_1, B_2, B_3, B_4 of formulas of $\mathcal{L}_{\{\Rightarrow,\cap\}}$ defined below determine if B_1, B_2, B_3, B_4 form a FORMAL PROOF in S.

If YES, provide comments how each step of the proof was obtained.

If NOT, write the reason why.

$$B_1 = (A \Rightarrow (B \Rightarrow A)),$$

$$B_2 = (B \Rightarrow (A \Rightarrow B)),$$

$$B_3 = ((B \Rightarrow (A \Rightarrow B)) \cap (A \Rightarrow (B \Rightarrow A))),$$

$$B_4 = ((A \Rightarrow (B \Rightarrow A)) \Rightarrow ((B \Rightarrow (A \Rightarrow B)) \cap (A \Rightarrow (B \Rightarrow A))))$$

Problem 3 Let H be the proof system defined in Problem 1.

(a) Prove the following: $A \vdash_H (A \Rightarrow A)$

(b) We know that $\vdash_H (\neg A \Rightarrow (A \Rightarrow B))$. Prove, that $\neg A, A \vdash_H B$.

Problem 4 Here are consecutive steps $B_1, ..., B_5$ in a proof of $(B \Rightarrow \neg\neg B)$ in H_2.

  The comments included are incomplete.

 Complete the comments by writing all details for each step of the proof. You have to write down **the proper substitutions and formulas** used at each step of the proof.

$B_1 = ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))$
  Axiom $A3$

$B_2 = (\neg\neg\neg B \Rightarrow \neg B)$
  Already proved fact: $\vdash_{H_2}(\neg\neg B \Rightarrow B)$

$B_3 = ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)$
  (MP)

$B_4 = (B \Rightarrow (\neg\neg\neg B \Rightarrow B))$
  Axiom $A1$

$B_5 = (B \Rightarrow \neg\neg B)$
  Already proved fact: $(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_2} (A \Rightarrow C)$