CSE541 EXERCISE 9(1)

QUESTION 1

H is the following proof system:

$$H = (\ \mathcal{L}_{\{\Rightarrow,\neg\}},\ \mathcal{F},\ AX = \{A1, A2, A3\},\ \ MP\ )$$

A1 $(B \Rightarrow (A \Rightarrow B))$,

A2 $(B \Rightarrow \neg\neg B)$,

A3 $(A \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B)))$,

A4 $(\neg A \Rightarrow (A \Rightarrow B))$,

MP (Rule of inference)

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

(1) Prove that H is SOUND under classical semantics.

(2) Prove that **The Main Lemma** of Chapter 9 HOLDS for H.

The Main Lemma states:

For any formula A and a variable assignment v, if $A^{'}$, B_1 , B_2, ..., B_n are corresponding formulas defined by 0.1, then

$$B_1, B_2, ..., B_n\ \vdash\ A'.$$

Definition 0.1 *Let A be a formula and $b_1, b_2, ..., b_n$ be all propositional variables that occur in A. Let v be variable assignment $v : VAR \longrightarrow \{T, F\}$. We define, for $A, b_1, b_2, ..., b_n$ and v a corresponding formulas $A',\ B_1, B_2, ..., B_n$ as follows:*

$$A' = \begin{cases} A & \text{if } v^*(A) = T \\ \neg A & \text{if } v^*(A) = F \end{cases}$$

$$B_i = \begin{cases} b_i & \text{if } v(b_i) = T \\ \neg b_i & \text{if } v(b_i) = F \end{cases}$$

for $i = 1, 2, ..., n$.

QUESTION 2 Here is a fragment of the proof of the Main Lemma (chapter 9).

Case: A **is** $(A_1 \Rightarrow A_2)$

If A is of the form $(A_1 \Rightarrow A_2)$ then A_1 and A_2 have less than n connectives and so by the inductive assumption we have $B_1, B_2, ..., B_n \vdash A_1^{'}$ and $B_1, B_2, ..., B_n \vdash A_2^{'}$, where $B_1, B_2, ..., B_n$ are formulas corresponding to the propositional variables in A.

Write down all steps that justify correctness of the above statement.

QUESTION 3 Give examples of 3 proof systems S_1, S_2, S_3 with some sets of axioms that make them complete with respect to classical semantics. Two of them must have languages containing more connectives then $\neg, \Rightarrow$.

Justify why and how you can carry the proof of completeness theorem for each of them.

QUESTION 4 Here is a more detailed version of our Proof 1 of the Completeness Theorem (as in new Chapter 9)

Assume that $\models A$. Let $b_1, b_2, ..., b_n$ be all propositional variables that occur in A, i.e. $A = A(b_1, b_2, ..., b_n)$. By the Main Lemma we know that, for any variable assignment v, the corresponding formulas $A^{'}$, B_1, B_2, ..., B_n can be found such that $B_1, B_2, ..., B_n \vdash A^{'}$.

Note here that $A^{'}$ of the definition is A, since since $\models A$.

Let $v : VAR \rightarrow \{T, F\}$ be any variable assignment, and

$$v_A : \quad \{b_1, b_2, ..., b_n\} \quad \rightarrow \{T, F\} \tag{1}$$

its restriction to the formula A. We get by the Main Lemma that the v_A (1) and the formula A define the appropriate formulas $B_1, B_2, ..., B_n$ such that

$$B_1, B_2, ..., B_n \vdash A. \tag{2}$$

The proof is based on a method of constructive elimination of all hypothesis $B_1, B_2, ..., B_n$ in 2) to finally construct the proof of A in S i.e. we prove that $\vdash A$.

Step 1: elimination of B_n. The form of B_n depends of the logical values assigned to b_n by the v_A (1), so we have to consider 2 cases: $v_A(b_n) = T$ or $v_A(b_n) = F$.

Case 1: $v_A(b_n) = T$ and then by definition $B_n = b_n$ and by the Main Lemma

$$B_1, B_2, ..., b_n \vdash A.$$

By Deduction Theorem we have that

$$B_1, B_2, ..., B_{n-1} \vdash (b_n \Rightarrow A). \tag{3}$$

Case 2: $v_A(b_n) = F$ and hence $B_n = \neg b_n$. By the Main Lemma

$$B_1, B_2, ...B_{n-1}, \neg b_n \vdash A.$$

By the Deduction Theorem we have that

$$B_1, B_2, ..., B_{n-1} \vdash (\neg b_n \Rightarrow A). \tag{4}$$

By the assumed provability of the formula 9 for $A = b_n, B = A$ we have that

$$\vdash ((b_n \Rightarrow A) \Rightarrow ((\neg b_n \Rightarrow A) \Rightarrow A)).$$

By monotonicity we have that

$$B_1, B_2, ..., B_{n-1} \vdash ((b_n \Rightarrow A) \Rightarrow ((\neg b_n \Rightarrow A) \Rightarrow A)). \tag{5}$$

Applying Modus Ponens twice to the above property 5 and properties (3), (4) we get that

$$B_1, B_2, ..., B_{n-1} \vdash A. \tag{6}$$

and hence we have eliminated B_n.

Step 2: elimination of B_{n-1}. We repeat the Step 1.
As before we have 2 cases to consider: $v_A(b_{n-1})$ may be T or F. In both cases we apply Main Lemma, Deduction Theorem, monotonicity, proper substitutions of assumed provability of the formula 9 i.e the fact that $\vdash ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$, and Modus Ponens twice and eliminate B_{n-1} just as we eliminated B_n.

After n steps, we finally obtain that

$$\vdash A.$$

Write down following the STEP 1, all detailed steps of the Step 2: elimination of B_{n-1}.

QUESTION 5 Apply the above proof step by step to construct proof in S of the following de Morgan Law.

$$(\neg(a \cup b) \Rightarrow (\neg a \cap \neg b)).$$

Hint the proof 1 works only for the language $\mathcal{L}_{\{\Rightarrow,\neg\}}$.