CSE541 EXERCISE 9

Problem 1

H is the following proof system:

$$H = (\ \mathcal{L}_{\{\Rightarrow\}},\ \mathcal{F},\ AX = \{A1, A2\},\ \ MP\ )$$

A1 $(B \Rightarrow (A \Rightarrow B))$,

A2 $((B \Rightarrow (A \Rightarrow C)) \Rightarrow ((B \Rightarrow A) \Rightarrow (B \Rightarrow C)))$,

MP (Rule of inference)

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

(1) Prove that H is SOUND under classical semantics.

(2) Does Deduction Theorem holds for H? Justify.

(3) Is H COMPLETE with respect to all classical semantics tautologies? Justify.

Problem 2 Here are consecutive steps $B_1, ..., B_9$ in a proof of

$$((B \Rightarrow A) \Rightarrow (\neg A \Rightarrow \neg B))$$

in H_2 of our Book. The comments included are incomplete.

Complete the comments by writing all details. You have to write down the proper substitutions and formulas used at each step of the proof.

$B_1 = (B \Rightarrow A)$
Hypothesis

$B_2 = (\neg\neg B \Rightarrow B)$
Already proved formula: $(\neg\neg A \Rightarrow A)$ for

$B_3 = (\neg\neg B \Rightarrow A)$
Already proved fact: $(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_2} (A \Rightarrow C)$

$B_4 = (A \Rightarrow \neg\neg A)$

$B_5 = \ (\neg\neg B \Rightarrow \neg\neg A)$

Already proved fact: $(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_2} (A \Rightarrow C)$

$B_6 = \ ((\neg\neg B \Rightarrow \neg\neg A) \Rightarrow (\neg A \Rightarrow \neg B))$

Already proved formula: $((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B))$ for

$B_7 = \ (\neg A \Rightarrow \neg B)$

$B_8 = \ (B \Rightarrow A) \vdash (\neg A \Rightarrow \neg B)$

$B_9 = \ ((B \Rightarrow A) \Rightarrow (\neg A \Rightarrow \neg B))$

Problem 3 Let H' be the proof system obtained from the system H of Problem 1 by extending the language to contain the negation $\neg$ and adding an additional axiom

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$

(a) Is the system H' complete? Does Deduction Theorem holds for H'? Justify.

(b) Prove the following: $A \vdash_{H'} (A \Rightarrow A)$

(c) We know that $\vdash_{H'} (\neg A \Rightarrow (A \Rightarrow B))$. Prove, that $\neg A, A \vdash_{H'} B$.

Problem 4 Let A be a formula

$$((\neg a \Rightarrow \neg b) \Rightarrow (c \cup a))$$

and let v be such that

$$v(a) = T, \quad v(b) = F, v(c) = F.$$

PART 1

Evaluate A', $B_1, ...B_n$ as defined by the following definition. Write carefully all steps. You can use shorthand notation.

Definition Let A be a formula and $b_1, b_2, ..., b_n$ be all propositional variables that occur in A. Let v be variable assignment $v : VAR \longrightarrow \{T, F\}$. We define, for $A, b_1, b_2, ..., b_n$ and v a corresponding formulas A', $B_1, B_2, ..., B_n$ as follows:

$$A' = \begin{cases} A & \text{if } v^*(A) = T \\ \neg A & \text{if } v^*(A) = F \end{cases}$$

$$B_i = \begin{cases} b_i & \text{if } v(b_i) = T \\ \neg b_i & \text{if } v(b_i) = F \end{cases}$$

for $i = 1, 2, ..., n$.

PART 2 The **Lemma** stated below describes a method of transforming a semantic notion of a tautology into a syntactic notion of provability. It defines, for any formula A and a variable assignment v a corresponding deducibility relation $\vdash$.

Lemma For any formula A and a variable assignment v, if $A^{'}$, B_1 , B_2, ..., B_n are corresponding formulas defined by the definition stated above, then

$$B_1, B_2, ..., B_n \vdash_{H2} A'.$$

1. STATE what the Lemma asserts for A, v be as defined above.

2. We prove the **Lemma** by induction on the structure of A i.e. by induction on the degree of the formula A.

Write the proof of the base case, i.e the case $n = 0$.

Write the proof of the case of B of the form $\neg A$.

Solve PROBLEMS listed at the end of section **Proof 1: Examples and Exercises** of Chapter 9, i.e. the following problems.

Problem 5

For the formulas A_i and corresponding truth assignments v find formulas $B_1, ..B_k, A_i^{'}$ as described by the Main Lemma, i.e. such that

$$B_1, ...B_k \vdash A_i^{'}.$$

1. $A_1 = ((\neg(b \Rightarrow a) \Rightarrow \neg a) \Rightarrow ((\neg b \Rightarrow (a \Rightarrow \neg c)) \Rightarrow c))$
$v(a) = T, v(b) = F, v(c) = T.$

2. $A_2 = ((a \Rightarrow (c \Rightarrow (\neg b \Rightarrow c))) \Rightarrow ((\neg d \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))) \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))))$
$v(a) = F, v(b) = F, v(c) = T, v(d) = F$

3. $A_3 = (\neg b \Rightarrow (c \Rightarrow (\neg a \Rightarrow b)))$
$v(a) = F, v(b) = F, v(c) = T$

4. $A_4 = (\neg a_1 \Rightarrow (a_2 \Rightarrow (\neg a_3 \Rightarrow a_1)))$
$v(a_1) = F, v(a_2) = F, v(a_3) = T$

5. $A_5 = ((b \Rightarrow (a_1 \Rightarrow (\neg c \Rightarrow b))) \Rightarrow ((\neg b \Rightarrow (a_2 \Rightarrow (\neg a_1 \Rightarrow b))) \Rightarrow (c \Rightarrow (\neg a \Rightarrow b)))$
$v(a) = F, v(b) = T, v(c) = F, v(a_1) = T, v(a_2) = F$

Problem 6 For any of the formulas A_1, A_2, A_3, A_4 listed below construct their proofs, as described in the proof 1 of the Completeness Theorem.

$A_1 = (\neg\neg b \Rightarrow b)$

$A_2 = ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a))$

$A_3 = (\neg(a \Rightarrow b) \Rightarrow \neg(\neg b \Rightarrow \neg a))$

$A_4 = ((a \Rightarrow (b \Rightarrow \neg a)) \Rightarrow (\neg(b \Rightarrow \neg a) \Rightarrow \neg a))$

For the Midterm you must know how to solve problems of theses two types..