cse547, math547 DISCRETE MATHEMATICS Lectures Content Final Infinite Series, Chapter 3 and Chapter 4

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#### CHAPTER 2 PART 5: INFINITE SUMS (SERIES)

Here are Definitions, Basic Theorems and Examples you must know

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#### Series

#### Definitions, Theorems, Simple Examples

Must Know STATEMENTS- do not need to PROVE the Theorems

#### Definition

If the limit  $\lim_{n\to\infty} S_n$  exists and is finite, i.e.

 $\lim_{n\to\infty}S_n=S,$ 

then we say that the infinite sum  $\sum_{n=1}^{\infty} a_n$  converges to S and we write

$$\Sigma_{n=1}^{\infty} a_n = \lim_{n \to \infty} \Sigma_{k=1}^n a_k = S,$$

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otherwise the infinite sum diverges

#### Show

The infinite sum  $\sum_{n=1}^{\infty} (-1)^n$  diverges

The infinite sum  $\sum_{n=0}^{\infty} \frac{1}{(k+1)(k+2)}$  converges to 1

#### Theorem 1

If the infinite sum

$$\sum_{n=1}^{\infty} a_n$$
 converges, then  $\lim_{n \to \infty} a_n = 0$ 

#### **Definition 5**

An infinite sum

$$\Sigma_{n=1}^{\infty}(-1)^{n+1}a_n$$
, for  $a_n \geq 0$ 

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is called **alternating infinite sum** (alternating series)

**Theorem 6** Comparing the series Let  $\sum_{n=1}^{\infty} a_n$  be an infinite sum and  $\{b_n\}$  be a sequence such that

 $0 \le b_n \le a_n$  for all n

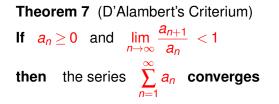
If the infinite sum  $\sum_{n=1}^{\infty} a_n$  converges then  $\sum_{n=1}^{\infty} b_n$  also converges and

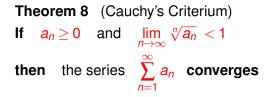
 $\Sigma_{n=1}^{\infty}b_n \leq \Sigma_{n=1}^{\infty}a_n$ 

Use Theorem 6 to prove that the series,

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$$

converges





Theorem 9 (Divergence Criteria)

If  $a_n \ge 0$  and  $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} > 1$  or  $\lim_{n \to \infty} \sqrt[n]{a_n} > 1$ then the series  $\sum_{n=1}^{\infty} a_n$  diverges Prove The series  $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$  does not react on D'Alambert's Criterium (Theorem 7)

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#### STUDY ALL EXAMPLES from Lecture 10

#### CHAPTER 3 INTEGER FUNCTIONS

## Here is the **proofs** in course material you need to know for **Midterm 2** and **Final** Plus the regular Homeworks Problems

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Prove the following Fact 3 For any  $x, y \in R$   $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$  when  $0 \le \{x\} + \{y\} < 1$ and

 $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor + 1 \quad \text{when} \quad 1 \le \{x\} + \{y\} < 2$ Fact 5 For any  $x \in \mathbb{R}, x \ge 0$  the following property holds  $\left| \sqrt{\lfloor x \rfloor} \right| = \lfloor \sqrt{x} \rfloor$ 

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Prove the following properties of characteristic functions **F1** For any predicates P(k), Q(k)

 $[P(k) \cap Q(k)] = [P(k)][Q(k)]$ 

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**F2** For any predicates P(k), Q(k)  $[P(k) \cup Q(k)] = [P(k)] + [Q(k)] - [P(k) \cap Q(k)]$ 

Prove the Combined Domains Property **Property 4** 

$$\sum_{Q(k)\cup R(k)}a_k=\sum_{Q(k)}a_k+\sum_{R(k)}a_k-\sum_{Q(k)\cap R(k)}a_k$$

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where, as before,

 $k \in K$  and  $K = K_1 \times K_2 \cdots \times K_i$  for  $1 \le i \le n$ and the above formula represents single (i = 1) and multiple (i > 1) sums

Study all 7 steps of our explanations to BOOK solution I will give you ONE to write in full on the test

$$1 \quad W = \sum_{n=1}^{1000} [n \text{ is a winner }] = \sum_{n=1}^{1000} [\lfloor \sqrt[3]{n} \rfloor \mid n]$$

$$2 \quad W = \sum_{k,n} [k = \lfloor \sqrt[3]{n} \rfloor] [k|n] [1 \le n \le 1000]$$

$$3 \quad W = \sum_{k,n,m} [k^3 \le n < (k+1)^3] [n = km] [1 \le n \le 1000]$$

$$4 \quad W = 1 + \sum_{k,m} [k^3 \le km < (k+1)^3] [1 \le k < 10]$$

$$5 \quad W = 1 + \sum_{k,m} \left[ m \in \left[k^2 \dots \frac{(k+1)^3}{k}\right] \right] [1 \le k < 10]$$

$$6 \quad W = 1 + \sum_{k,m} \left[ m \in \left[k^2 \dots \frac{(k+1)^3}{k}\right] \right] [1 \le k < 10]$$

$$6 \quad W = 1 + \sum_{1 \le k < 10} \left( \lceil k^2 + 3k + 3 + \frac{1}{k} \rceil - \lceil k^2 \rceil \right)$$

$$7 \quad W = 1 + \sum_{1 \le k < 10} (3k + 4) = 1 + \frac{7 + 31}{2}9 = 172$$

#### PART2: Spectrum Partitions

Prove the following properties

P1 
$$\sum_{k} [R(k)] = \sum_{R(k)} 1 = |R(k)|$$
  
P2  $\sum_{k,m} [P(m)] [Q(k)] = \sum_{Q(k)} \sum_{P(m)} 1 = \sum_{Q(k)} |P(m)|$ 

where we denote for short

$$| P(m) | = | \{ m \in N : P(m) \} |$$

Justify that

$$N(\alpha,n)=\sum_{k>0}\left[k<\frac{n+1}{\alpha}\right]$$

Write a detailed proof of

$$N(\alpha,n) = \left\lceil \frac{n+1}{\alpha} \right\rceil - 1$$

Write a detailed proof of

#### PART2: Spectrum Partitions

### **Prove** the following **Fact P2** If |A| + |B| = |X| and $A \neq \emptyset$ , $B \neq \emptyset$ and $A \cap B = \emptyset$ then the sets A, B form a finite partition of XSpectrum Fact

$$Spec(\sqrt{2}) \cap Spec(2+\sqrt{2}) = \emptyset$$

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#### **Finite Spectrum Partition Theorem**

- **1.**  $A_n \neq \emptyset$  and  $B_n \neq \emptyset$
- **2.**  $A_n \cap B_n = \emptyset$
- **3.**  $A_n \cup B_n = \{1, 2, \dots, n\}$

#### PART2: Spectrum Partitions

**Prove** - use your favorite proof out of the two I have provided

#### **Spectrum Partition Theorem**

- **1.** Spec( $\sqrt{2}$ )  $\neq \emptyset$  and Spec( $2 + \sqrt{2}$ )  $\neq \emptyset$
- **2.** Spec( $\sqrt{2}$ )  $\cap$  Spec( $2 + \sqrt{2}$ ) =  $\emptyset$
- 3. Spec $(\sqrt{2}) \cup$  Spec $(2 + \sqrt{2}) = N \{0\}$

#### PART3: Sums

Write detailed evaluation of

 $\sum_{0 \le k < n} \lfloor \sqrt{k} \rfloor$ 

Hint: use

$$\sum_{0 \le k < n} \lfloor \sqrt{k} \rfloor = \sum_{0 \le k < n} \sum_{m \ge 0, m = \lfloor \sqrt{k} \rfloor} m$$

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#### Chapter 4 Material in the Lecture 12

**JUSTIFY** correctness of the following example and be ready to do similar problems upwards or downwards Represent 19151 in a system with base 12 **Example** 

 $19151 = 1595 \cdot 12 + 11$   $1595 = 132 \cdot 12 + 11$   $132 = 11 \cdot 12 + 0$   $a_0 = 11, \quad a_1 = 11, \quad a_2 = 0, \quad a_3 = 11$ So we get

 $19151 = (11, 0, 11, 11)_{12}$ 

# Write a proof of Step 1 or Step 2 of the Proof of the Correctness of Euclid Algorithm

You can use Lecture OR BOOK formalization and proofs

#### Use Euclid Algorithms to prove

When a product ac of two natural numbers is divisible by a number b that is **relatively prime** to a, the factor c must be divisible by b

Use Euclid Algorithms to prove the following Fact

 $gcd(ka,kb) = k \cdot gcd(a,b)$ 

#### Prove:

Any common multiple of **a** and **b** is **divisible** by lcm(a,b) **Prove** the following

$$\forall_{p,q_1q_2\dots q_n \in P} (p \mid \prod_{k=1}^n q_k \Rightarrow \exists_{1 \leq i \leq n} (p = q_i))$$

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Write down a formal formulation (in all details ) of the Main Factorization Theorem and its General Form

**Prove** that the representation given by Main Factorization Theorem is unique

Explain why and show that 18 = < 1, 2 >

#### Prove

 $k = gcd(m, n) \quad \text{if and only if} \quad k_p = min\{m_p, n_p\}$   $k = lcd(m, n) \quad \text{if and only if} \quad k_p = max\{m_p, n_p\}$ Let  $m = 2^0 \cdot 3^3 \cdot 5^2 \cdot 7^0 \qquad n = 2^0 \cdot 3^1 \cdot 5^0 \cdot 7^3$ Evaluate gcd(m, n) and k = lcd(m, n)

#### Exercises

1. Use Facts 6-8 to prove

#### Theorem 5

For any  $a, b \in Z^+$  such that lcm(a,b) and gcd(a, b) exist

 $lcm(a,b) \cdot gcd(a,b) = ab$ 

**2.** Use **Theorem 5** and the BOOK version of Euclid Algorithm to express  $lcm(n \mod m, m)$  when  $nmodm \neq 0$  This is Ch4 Problem 2

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