#### **Proof of Partition Theorem**

Spec( $\sqrt{2}$ ) and Spec(2 +  $\sqrt{2}$ ) Form a Partition of N-{0} = Z+

### Method : Prove General Theorem of which our theorem is a particular case

#### **SPECTRUM PARTITION THEOREM**

Let  $\alpha$ ,  $\beta > 0$ ,  $\alpha$ ,  $\beta \in \mathbb{R}$ -Q

Be such that  $1/\alpha + 1/\beta = 1$ 

#### Then the sets

A = { 
$$|\alpha n|$$
 : n = 1,2,3,....} = spec( $\alpha$ )  
A = {  $|\beta n|$  : n = 1,2,3,....} = spec( $\beta$ )

Form a Partition of Z+ = N - {0}

i.e. A≠φ, B≠φ A∩B = φ AUB = Z+

#### **Proof of Partition Theorem (Special Case)**

 $α = \sqrt{2}, β = 2 + \sqrt{2}$ 

We get :

## Spec( $\sqrt{2}$ ) and Spec(2 + $\sqrt{2}$ ) Form a Partition of N-{0} = Z+

Proof

- **1.** The  $[\alpha] \in A, [\beta] \in B$ **2.**  $A \cap B = \phi$
- **Proof by contradiction**
- Suppose that  $A \cap B \neq \phi$  i.e.
- i.e. There is k  $\in$  Z+ such that k  $\in$  A, k  $\in$  B
- Iff there are i,  $j \in Z$ + such that  $\lfloor \alpha i \rfloor = k \quad \lfloor \beta j \rfloor = k \quad i.e.$

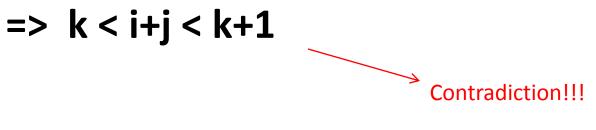
 $k \le \alpha i < k+1$  $k \le \beta j < k+1$ 

#### But $\alpha$ , $\beta \in R-Q$ ,

So αi, βj cant be integers, so ≤ cant hold, so we get k < αi < k+1 k < βj < k+1 k/α < i < (k+1) /α k/β < j < (k+1) /β

#### On adding them

## $k/\alpha + k/\beta < i + j < (k+1)/\alpha + (k+1)/\beta$ $k(1/\alpha + 1/\beta) < i + j < (k+1)(1/\alpha + 1/\beta)$ We know that $1/\alpha + 1/\beta = 1$



i,j,k € Z+

#### No integer between k, k+1!

We proved  $A \cap B = \phi$ 

Now we want to prove that AUB = Z+ Assume AUB  $\neq Z$ +

i.e. Exists k ∈ Z+, such that k ∉ AUB
i.e. k ∉ A and k ∉ B

k  $\notin$  A iff for all x  $\in$ Z+k  $\neq \lfloor \alpha n \rfloor$ -----1k  $\notin$  B iff for all x  $\in$ Z+k  $\neq \lfloor \beta n \rfloor$ -----2

This means that there exist  $i_0$ ,  $j_0$  such that  $[\alpha i_0] < k$  and  $[\alpha (i_0+1)] > k$  and same holds for  $\beta$ 

i.e.

## (1a) $\alpha i_0 < k \& \alpha (i_0+1) > k+1$ (2a) $\beta i_0 < k \& \beta (i_0+1) > k+1$ (These cant be = k+1 as $\beta, \alpha \in \mathbb{R}$ -Q and k+1 $\in \mathbb{Z}$ +) $\lfloor \beta i_0 \rfloor < k$ and $\beta (i_0+1) \rfloor > k$

## 1a when rewritten

- $\alpha < k/i_0 \& \alpha > (k+1)/(i_0+1)$
- $=> 1/\alpha > i_0/k \& 1/\alpha < (i_0+1)/(k+1)$
- Or  $i_0/k < 1/\alpha < (i_0+1)/(k+1) || || v for \beta we get$  $<math>j_0/k < 1/\beta < (j_0+1)/(k+1)$

# Adding the above two equations and using $1/\alpha + 1/\beta = 1$ We get

$$(i_0+j_0)/k < 1 < (i_0+j_0+2)/(k+1)  $\Rightarrow (i_0+j_0)/k < 1 \text{ and } 1 < (i_0+j_0+2)/(k+1)  $\Rightarrow i_0+j_0 < k \text{ and } k < i_0+j_0+1 \Rightarrow i_0+j_0 < k < i_0+j_0+1$  k, i_0,j_0  $\in Z+$$$$

#### **Contradiction:** n < k < n+1