Discrete Mathematics

Chapter 1, Problem 20

Use the repertoire method to solve the general five-parameter recurrence (R):

$h(1) = \alpha;$

$h(2n) = 4h(n) + \gamma_0 n + \beta_0;$

$h(2n+1) = 4h(n) + \gamma_1 n + \beta_1$, for $n \geq 1$.

General Guess

Solution:

- Repertoire method: examine a repertoire of cases and use results to find CF.

- We have 5 unknown parameters:
  α, γ_0, β_0, γ_1, $\beta_{1.}$

- Therefore, the general guess is:
  CF: $h(n) = \alpha A(n) + \beta_0 B(n) + \beta_1 C(n) + \gamma_0 D(n) + \gamma_1 E(n)$

Observing Pattern

- $h(1) = \alpha$; $h(2n) = 4h(n) + \beta_0$; $h(2n+1) = 4h(n) + \beta_1$, for $\gamma_0 = \gamma_1 = 0$, $n \geq 1$

We guess:

1. $A(n) = 4^m \rightarrow A(2^m+l) = 4^m$
2. $\beta_0 B(n) + \beta_1 C(n) = (\beta_{bm-1} \beta_{bm-2} \ldots \beta_{b1} \beta_{b0})_4$

Here if:

- bit b_j of n is 0 $\Rightarrow \beta_{bj} = \beta_0$
- bit b_j of n is 1 $\Rightarrow \beta_{bj} = \beta_1$

$\Rightarrow h(n) = \alpha A(n) + \beta_0 B(n) + \beta_1 C(n) = (\alpha \beta_{bm-1} \beta_{bm-2} \ldots \beta_{b1} \beta_{b0})_4$

n	h(n)
1	α
2	$4\alpha + \beta_0$
3	$4\alpha + \beta_1$
4	$16\alpha + 4\beta_0 + \beta_0$
5	$16\alpha + 4\beta_0 + \beta_1$
6	$16\alpha + 4\beta_1 + \beta_0$
7	$16\alpha + 4\beta_1 + \beta_1$
8	$64\alpha + 16\beta_0 + 4\beta_0 + \beta_0$
…	

Radix representation

- CF: $h(n) = \alpha A(n) + \beta_0 B(n) + \beta_1 C(n)$
  R: $h(1) = \alpha$; $h(2n) = 4h(n) + \beta_0$;
  $h(2n+1) = 4h(n) + \beta_1$, for $\gamma_0 = \gamma_1 = 0$, $n \geq 1$

- We apply (1.18) to start with numbers in radix d and produce numbers in radix c:
  - $f(i) = \alpha_i$, for $1 \leq i < d$
  - $f(dn+j) = cf(n) + \beta_j$, for $0 \leq j < d$ and $n \geq 1$
  - $f((b_m b_{m-1} b_{m-2} \ldots b_1 b_0)_d) = (\alpha \beta_{bm-1}\, \beta_{bm-2} \ldots \beta_{b1}\, \beta_{b0})_c$
- In our case: $d = 2$, $c = 4$.
  - $n = (1\, b_{m-1}\, b_{m-2} \ldots b_1\, b_0)_2$
  - $h((1\, b_{m-1}\, b_{m-2} \ldots b_1\, b_0)_2) = (\alpha \beta_{bm-1}\, \beta_{bm-2} \ldots \beta_{b1}\, \beta_{b0})_4$
- Fact 0:

$$\alpha A(n) + \beta_0 B(n) + \beta_1 C(n) = (\alpha \beta_{bm-1}\, \beta_{bm-2} \ldots \beta_{b1}\, \beta_{b0})_4$$

Step 1

- CF: $h(n) = \alpha A(n) + \beta_0 B(n) + \beta_1 C(n)$
  $+ \gamma_0 D(n) + \gamma_1 E(n)$

  R: $h(1) = \alpha$;

  $h(2n) = 4h(n) + \gamma_0 n + \beta_0$;

  $h(2n+1) = 4h(n) + \gamma_1 n + \beta_1$, for $n \geq 1$

- Consider case: $\alpha = 1$, $\beta_0 = \beta_1 = \gamma_0 = \gamma_1 = 0$;

  CF: $h(n) = A(n)$

  R: $h(1) = 1$; $h(2n) = 4h(n)$; $h(2n+1) = 4h(n)$;

  => $A(1) = 1$; $A(2n) = 4A(n)$; $A(2n+1) = 4A(n)$;

- Fact 1: $A(n) = 4^k$ for $n = 2^k + l$, $0 \leq l < 2^k$

Step 1 – Proof by induction

- $A(1) = 1;$

  $A(2n) = 4A(n);$

  $A(2n+1) = 4A(n);$
- $A(n) = 4^k$ for $n = 2^k + l, 0 \le l < 2^k$

- Base case: $k = 0 \Rightarrow n = 4^0 + l, 0 \le l < 1 \Rightarrow n = 1$
  - $h(n) = A(n)$ and $h(1) = 1 \Rightarrow A(1) = 1$
  - $A(1) = 4^0 = 1$ YES

Step 1 – Proof by induction

- $A(1) = 1;$
  $A(2n) = 4A(n);$
  $A(2n+1) = 4A(n);$
- $A(n) = 4^k$ for $n = 2^k + l,\ 0 \le l < 2^k$

- Assume: $A(2^{k-1} + l) = 4^{k-1},\ 0 \le l < 2^{k-1}$
- Prove: $A(2^k + l) = 4^k,\ 0 \le l < 2^k$
- Cases: n even, n odd
- 1) $n = 2n_1 \Rightarrow 2^k + l = 2n_1$ iff l even
  $\Rightarrow n_1 = 2^{k-1} + l/2$
  $A(2n_1) = A(2^k + l) = 4A(n_1)$
  $= 4A(2^{k-1} + l/2) = 4 * 4^{k-1} = 4^k$ YES

Step 1 – Proof by induction

- $A(1) = 1;$
  $A(2n) = 4A(n);$
  $A(2n+1) = 4A(n);$
- $A(n) = 4^k$ for $n = 2^k + l,\ 0 \le l < 2^k$

- Assume: $A(2^{k-1} + l) = 4^{k-1},\ 0 \le l < 2^{k-1}$
- Prove: $A(2^k + l) = 4^k,\ 0 \le l < 2^k$
- Cases: n even, n odd
- 2) $n = 2n_1+1 \Rightarrow 2^k + l = 2n_1+1$ iff l odd

$$\Rightarrow n_1 = 2^{k-1} + (l-1)/2$$

$$A(2n_1+1) = A(2^k + l) = 4A(n_1)$$

$$= 4A(2^{k-1} + (l-1)/2) = 4 * 4^{k-1} = 4^k \text{ YES}$$

Step 2

- CF: $h(n) = \alpha A(n) + \beta_0 B(n) + \beta_1 C(n) + \gamma_0 D(n) + \gamma_1 E(n)$
  R: $h(1) = \alpha$;
  $h(2n) = 4h(n) + \gamma_0 n + \beta_0$;
  $h(2n+1) = 4h(n) + \gamma_1 n + \beta_1$, for $n \geq 1$

- Consider the "constant" function:
  $h(n) = 1$ for all $n \geq 1$
- We plug this in our recurrence (R):
  $1 = \alpha$;
  $1 = 4*1 + \gamma_0 n + \beta_0 \rightarrow -3 = \gamma_0 n + \beta_0$
  $\rightarrow \beta_0 = -3, \gamma_0 = 0$;
  $1 = 4*1 + \gamma_1 n + \beta_1 \rightarrow -3 = \gamma_1 n + \beta_1$
  $\rightarrow \beta_1 = -3, \gamma_1 = 0$;

Fact 2: $A(n) – 3B(n) – 3C(n) = 1$, for $n \geq 1$

Step 3

- CF: $h(n) = \alpha A(n) + \beta_0 B(n) + \beta_1 C(n) + \gamma_0 D(n) + \gamma_1 E(n)$

  R: $h(1) = \alpha$;

  $h(2n) = 4h(n) + \gamma_0 n + \beta_0$;

  $h(2n+1) = 4h(n) + \gamma_1 n + \beta_1$, for $n \geq 1$

- Consider the "constant" function:

  $h(n) = n$ for all $n \geq 1$

- We plug this in our recurrence (R):

  $1 = \alpha$;

  $2n = 4n + \gamma_0 n + \beta_0 \rightarrow 0 = 2n + \gamma_0 n + \beta_0$

  $\rightarrow \beta_0 = 0, \gamma_0 = -2$;

  $2n+1 = 4n + \gamma_1 n + \beta_1 \rightarrow 1 = 2n + \gamma_1 n + \beta_1$

  $\rightarrow \beta_1 = 1, \gamma_1 = -2$;

  Fact 3: $A(n) + C(n) - 2D(n) - 2E(n) = n$, for $n \geq 1$

Step 4

- CF: $h(n) = \alpha A(n) + \beta_0 B(n) + \beta_1 C(n) + \gamma_0 D(n) + \gamma_1 E(n)$
  R: $h(1) = \alpha$;
  $h(2n) = 4h(n) + \gamma_0 n + \beta_0$;
  $h(2n+1) = 4h(n) + \gamma_1 n + \beta_1$, for $n \geq 1$

- Consider the "constant" function:
  $h(n) = n^2$ for all $n \geq 1$

- We plug this in our recurrence (R):
  $1 = \alpha$;
  $4n^2 = 4n^2 + \gamma_0 n + \beta_0 \rightarrow 0 = \gamma_0 n + \beta_0$
  $\rightarrow \beta_0 = 0, \gamma_0 = 0$;
  $4n^2 + 4n + 1 = 4n^2 + \gamma_1 n + \beta_1 \rightarrow 4n + 1 = \gamma_1 n + \beta_1$
  $\rightarrow \beta_1 = 1, \gamma_1 = 4$;

Fact 4: $A(n) + C(n) + 4E(n) = n^2$, for $n \geq 1$

Final step

- Fact 0: $\alpha A(n) + \beta_0 B(n) + \beta_1 C(n) = (\alpha\beta_{bm-1}\ldots\beta_{b1}\,\beta_{b0})_4$
- Fact 1: $A(n) = 4^k$ for $n = 2^k + l$, $0 \leq l < 2^k$
- Fact 2: $A(n) – 3B(n) – 3C(n) = 1$
- Fact 3: $A(n) + C(n) – 2D(n) – 2E(n) = n$
- Fact 4: $A(n) + C(n) + 4E(n) = n^2$

- 1 => $A(n) = 4^k$ for $n = 2^k + l$, $0 \leq l < 2^k$
- 1 & 2 => $B(n) = ((1- 4^k)+ 3C(n)) / 3$ (5)
- 1 & 4 => $E(n) = (n^2 - 4^k - C(n)) / 4$ (6)
- 1 & 3 & 6 => $D(n) = (4^k + C(n) – 2E(n) - n) / 2$

  => $D(n) = (4^k + C(n) – 2((n^2 - 4^k - C(n)) / 4) - n) / 2$

  => $D(n) = ((3*4^k)/2 + 3*C(n)/2 – n^2/2 - n) / 2$ (7)
- 0 & 1 & 5

  => $\alpha\, 4^k + \beta_0(((1- 4^k)+ 3C(n)) / 3) + \beta_1 C(n) = (\alpha\beta_{bm-1}\ldots\beta_{b1}\,\beta_{b0})_4$

  => $C(n) = ((\alpha\beta_{bm-1}\ldots\beta_{b1}\,\beta_{b0})_4 - \alpha\, 4^k - \beta_0(1- 4^k)/3\ ) / (\beta_0 + \beta_1)$ (8)