CSE 547
Chapter 1, problem 20

Solution 2

Chapter 1 problem 20

Solve the general five-parameter recurrence

- $h(1)=\alpha$;
- $h(2n+j) = 4h(n) + \gamma_j n + \beta_j$, for $j=0,1$ and $n \geq 1$ for $n \in N$

According to the book, we should ideally solve $h(n)$ for 5 parameters, and find the general equation

$$h(n) = a(n)\alpha+b(n)\beta_0+c(n)\beta_1+d(n)\gamma_0+e(n)\gamma_1.$$

Instead, we're going to take a different approach, using binary numbers.

First 5 solutions:

$h(1) = \alpha$

$h(2) = h(2*1+0) = 4h(1)+n\gamma_0+\beta_0 = 4\alpha+\gamma_0+\beta_0$

$h(3) = h(2*1+1) = 4h(1)+n\gamma_1+\beta_1 = 4\alpha+\gamma_1+\beta_1$

$h(4) = h(2*2+0) = 4h(2)+n\gamma_0+\beta_0 =$
$4(4\alpha+\gamma_0+\beta_0)+2\gamma_0+\beta_0 =$
$16\alpha+4\gamma_0+4\beta_0+2\gamma_0+\beta_0 = 16\alpha+6\gamma_0+5\beta_0$

$h(5) = h(2*2+1) = 4h(2)+n\gamma_1+\beta_1 =$
$4(4\alpha+\gamma_0+\beta_0) + 2\gamma_1+\beta_1 = 16\alpha+4\gamma_0+4\beta_0+$
$2\gamma_1+\beta_1$

There is no obvious pattern from these steps

We can use binary numbers to find the solution.

We can write out the number k in binary expansion:

$k=(1\ b_{m-1}\ b_{m-2}\ldots b_1\ b_0)_2$

We know that each term b_i is 0 or 1, with the leading digit, α, always 1.

$h(1) = \alpha$

$h(2n+j) = 4h(n) + \gamma_j n + \beta_j$ for j=0,1

$h(1\ b_{m-1}b_{m-2}\ldots b_1 b_0)_2 = 4h(1\ b_{m-1}b_{m-2}\ldots b_1)_2 + \gamma_{b_0}(1\ b_{m-1}b_{m-2}\ldots b_1)_2 + \beta_{b_0}$

$= 4[4h(1\ b_{m-1}b_{m-2}\ldots b_2)_2 + \gamma_{b_m}(1\ b_{m-1}b_{m-2}\ldots b_2)_2 + \beta_{b_1}] + \gamma_{b_1}(1\ b_{m-1}b_{m-2}\ldots b_2)_2 + \beta_{b_1} + \gamma_{b_0}(1\ b_{m-1}b_{m-2}\ldots b_1)_2 + \beta_{b_0}$

As we can see, by expanding the formula, we reduce the number of digits inside each binary term

bAs we expand the formula completely, the equation becomes

$h(1\ b_{m-1}b_{m-2}\ldots b_1b_0)_2 = 4^m\alpha + 4^{m-1}(1)_2\gamma_{b_{m-1}} + 4^{m-2}(1\ b_{m-2})_2\gamma_{b_{m-2}} \ldots + 4^0(1\ldots b_1)_2\gamma_{b_0} + 4^{m-1}\beta_{b_{m-1}} + 4^{m-1}\beta_{b_{m-2}} \ldots + 4^1\beta_{b_1} + 4^0\beta_{b_0}$

Where m+1 is the number of digits in the binary number $k = (1\ b_{m-1}b_{m-2}\ldots b_1b_0)_2 = (b_m\ldots b_o)_2$.

$h(1\ b_{m-1}b_{m-2}\ldots b_1b_0)_2 = 4^m\alpha + 4^{m-1}(1)_2\gamma_{b_{m-1}} + 4^{m-2}(1\ b_{m-2})_2\gamma_{b_{m-2}} \ldots + 4^0(1\ldots b_1)_2\gamma_{b_0} + 4^{m-1}\beta_{b_{m-1}} + 4^{m-1}\beta_{b_{m-2}} \ldots + 4^1\beta_{b_1} + 4^0\beta_{b_0}$

We can combine all the γ terms as a summation, which becomes

$\sum 4^{m-i}(1\ldots b_{m-i+1})_2\gamma_{b_{m-i}}$

summation for $i: 1 \le i \le m$.

Similarly we can do the same thing to all the β terms, which turns into

$\sum 4^{m-i}\,\beta_{b_{m-I}}$,

summation for $i: 1 \le i \le m$

m+1 is the number of digits of the binary number $k = (1\ b_{m-1}b_{m-2}\ldots b_1b_0)_2 = (b_m\ldots b_o)_2$

The closed formula becomes

$h(b_m \ldots b_o) = 4^m\alpha + \sum 4^{m-i}(1\ldots b_{m-i+1})_2 \gamma_{b_{m-i}} + \sum 4^{m-i}\, \beta_{b_{m-l}}$

wbbhere summation is for i: $1 \le i \le m$, m+1 is the number of digits in the binary number

$k = (1\ b_{m-1} b_{m-2} \ldots b_1 b_0)_2 = (b_m \ldots b_o)_2\ h(b_m \ldots b_o)$, Lets try k=5, which we solved already

$h(5) = 16\alpha+4\gamma_0+4\beta_0+ 2\gamma_1+\beta_1$

In binary form: $h(5) = (101)$

$h(5) = 4^m\alpha+\sum 4^{m-i}(1\ldots b_{m-i+1})_2\gamma_{b_{m-i}} + \sum 4^{m-i}\,\beta_{b_{m-l}}$

$h(5) = 4^2\alpha+4^1(1)\gamma_{b_1} + 4^0(10)\gamma_{b_0} + 4^1\beta_{b_1} +4^0\beta_{b_0}$

$h(5) = 16\alpha +4\gamma_{b_1} + 1(2)\gamma_{b_0} + 4\beta_{b_1} +\beta_{b_0}$

$h(5) = 16\alpha +4\gamma_0 + 2\gamma_1 + 4\beta_0 +\beta_1$

$h(1) = 4^0\alpha + 0\gamma_{b_{m-i}} + 0\beta_{b_{m-i}} = \alpha$

$h(2) = 4^1\alpha + 4^0(1)\gamma_{b_0} + \beta_{b_0} = 4\alpha + \gamma_0 + \beta_0$

$h(3) = 4^1\alpha + 4^0(1)\gamma_{b_0} + 4^0\beta_{b_0} = 4\alpha + \gamma_1 + \beta_1$

$h(4) = 4^2\alpha + 4^1(1)\gamma_{b_1}+4^0(10)\gamma_{b_0}+4^1\beta_{b_1}+4^0\beta_{b_0}$

$= 16\alpha + 4(1)\gamma_0 + (2)\gamma_0 + 4\beta_0 +\beta_0$

$= 16\alpha + 6\gamma_0 + 5\beta_0$

Which are the same values we got recursively.