Cse547
Chapter 1, problem 9

Problem9

Sometimes it's possible to use induction backwards, proving things from n to n-1 instead of vice versa! For example, consider the statement

$$P(n): x_1 \ldots x_n \leq \left(\frac{x_1 + \ldots + x_n}{n}\right)^n, if x_1, \ldots, x_n \geq 0.$$

This is true when n $=$ 2, since $(x_1 + x_2)^2 - 4x_1x_2 = (x_1 - x_2)^2 \leq 0$.

Problem9

a By setting $x_n = (x_1 + x_2 + \ldots + x_n)/(n-1)$, prove that P(n) implies P(n-1) whenever $n > 1$.

b Show that P(n) and P(2) imply P(2n).

c Explain why this implies the truth of P(n) for all n.

(a)

- If we want to get P(n-1) from P(n), we have to eliminate xn by constant or x1…xn-1.
- If using constant, the left seems good, the right hand side is hard to go on: the constant c is hard to deal with

$$\left(\frac{x_1+x_2+\cdots+x_{n-1}+c}{n}\right)^n$$

(a)

- Then we try to swap xn with expression of x1…xn-1. Intuitively, it should be symmetrical.
- It is not hard to guess out the hint:

$$x_n = (x_1 + x_2 + \ldots + x_{n-1})/(n-1)$$

(a)

- By substituting xn by $x_n = (x_1 + x_2 + \ldots + x_{n-1})/(n-1)$

  We have:

$$x_1 x_2 \ldots x_n \leq \left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right)^n \qquad P(n)$$

$$\Rightarrow x_1 x_2 \ldots x_{n-1} \frac{x_1 + x_2 + \ldots + x_{n-1}}{n-1} \leq \left(\frac{x_1 + x_2 + \ldots + x_{n-1} + \frac{x_1+x_2+\ldots+x_{n-1}}{n-1}}{n}\right)^n$$

$$\Rightarrow x_1 x_2 \ldots x_{n-1} \frac{x_1 + x_2 + \ldots + x_{n-1}}{n-1} \leq \left(\frac{\frac{(n-1)(x_1+x_2+\ldots+x_{n-1})}{n-1} + \frac{x_1+x_2+\ldots+x_{n-1}}{n-1}}{n}\right)^n$$

$$\Rightarrow x_1 x_2 \ldots x_{n-1} \frac{x_1 + x_2 + \ldots + x_{n-1}}{n-1} \leq \left(\frac{x_1 + x_2 + \ldots + x_{n-1}}{n-1}\right)^n$$

$$\Rightarrow x_1 x_2 \ldots x_{n-1} \leq \left(\frac{x_1 + x_2 + \ldots + x_{n-1}}{n-1}\right)^{n-1} \qquad P(n-1)$$

(b)

- If we want to prove P(2n), we should first have the elements x1…x2n.
- First guess is to use P(n) twice, like this:

$$x_1 x_2 \ldots x_n \le \left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right)^n \qquad \textit{Using } P(n)$$

$$x_{n+1} x_{n+2} \ldots x_{2n} \le \left(\frac{x_{n+1} + x_{n+2} + \ldots + x_{2n}}{n}\right)^n \qquad \textit{Using } P(n)$$

- Then multiply them

(b)

- Then by using P(2), we can have:

$$x_1x_2\ldots x_{2n}$$

$$\leq \left(\frac{x_1+x_2+\ldots+x_n}{n}\cdot\frac{x_{n+1}+x_{n+2}+\ldots+x_{2n}}{n}\right)^n$$

$$\leq \left(\left(\frac{\frac{x_1+x_2+\ldots+x_n+x_{n+1}+x_{n+2}+\ldots+x_{2n}}{n}}{2}\right)^2\right)^n \qquad \textit{Using } P(2)$$

$$= \left(\frac{x_1+x_2+\ldots+x_{2n}}{2n}\right)^{2n}$$

(b)

- The About method uses P(n), then P(2).
- What if we use P(2), then P(n)?
- The method is to aggregate x(2i-1) and x(2i) into one part:

(b)

$$
\begin{aligned}
& x_1 x_2 \ldots x_{2n} \\
& \leq \left(\frac{x_1 + x_2}{2}\right)^2 \left(\frac{x_3 + x_4}{2}\right)^2 \ldots \left(\frac{x_{2n-1} + x_{2n}}{2}\right)^2 && \textit{Using } P(2) \\
& \leq \left(\left(\frac{\frac{(x_1+x_2)+(x_3+x_4)+\ldots+(x_{2n-1}+x_{2n})}{2}}{n}\right)^2\right)^n && \textit{Using } P(n) \\
& = \left(\frac{x_1 + x_2 + \ldots + x_{2n}}{2n}\right)^{2n}
\end{aligned}
$$

(c)

- The idea is to use a special version of mathematical induction as this form:

- Basis: the statement holds when n=1or2.
- Inductive: if the statement holds for n<=k,

  then the same statement holds

  for n=k+1

(c)

- Basis: for n=1, $x_1 \cdot (x_1/1)^1$.

  So we have P(1)

  for n=2, $x_1 + x_2 \cdot ((x_1 + x_2)/2)^2$.

  So we have P(2)

(c)

- Inductive step:

  Suppose for n<=k, we have P(n)

  then for n=k+1, we do a case analysis to

  show P(n) still holds:

(c)

- if k+1 is even, then (k+1)/2 is an integer and less than k+1, So P((k+1)/2) holds.

  Based on Question(b), P((k+1)/2)&P(2)=>P(k+1)

- If k+1 is odd, then (k+2)/2 is an integer and less than k+1, so P((k+2)/2) holds.

  Based on Question(b), P((k+2)/2)&P(2)=>P(k+2).

  Based on Question(a), P(k+2)=>P(k+1)

(c)

- Therefore, for n=k+1, P(n) holds for all cases. Proof is done.