CSE 547

Chapter 2, Problem 15

Problem Statement

Evaluate by text Method 5

$$\blacksquare_n = \sum_{k=1}^{n} k^3$$

Hint: First prove

$$\blacksquare_n + \square_n = 2\sum_{1 \le j \le k \le n} jk$$

then Apply Equation(2.33)

Sum of the first 10 cubes

n	1	2	3	4	5	6	7	8	9	10	…
N^3	**1**	**8**	**27**	**64**	**125**	**216**	**343**	**512**	**729**	**1000**	**…**
$\blacksquare_n$	**1**	**9**	**36**	**100**	**225**	**441**	**784**	**1296**	**2025**	**3025**	**…**

As evident we cannot find a closed form for $\blacksquare_n$. But when we do find one we can use these values for verification.

Review Method 5: Expand and Contract

Finding a closed form for the sum of the first n squares,

$$\square_n = \sum_{0 \le k \le n} k^2 \text{, for n >= 0.}$$

In method 5, described in page 46 of the text book, we replace the original sum by a seemingly more complicated multiple sum that can actually be simplified if we operate on them properly :

Method 5 cont (look lecture notes for details!)…

$$\square_n = \sum_{1\le k\le n} k^2$$

$$= \sum_{1\le j\le k\le n} k \quad = \sum_{1\le j\le n} \sum_{j\le k\le n} k$$

$$= \sum_{1\le j\le n} \left(\frac{j+n}{2}\right)(n-j+1), \qquad \text{Average} = \frac{\text{first \# + last \#}}{2}$$

$$\text{\# numbers} = (\text{last \# - first \# + 1})$$

$$= \frac{1}{2}\sum_{1\le j\le n}\left(n(n+1)+j-j^2\right)$$

$$= \frac{1}{2}n^2(n+1)+\frac{1}{4}n(n+1)-\frac{1}{2}\sum_{1\le j\le n} j^2$$

$$= \frac{1}{2}n^2(n+1)+\frac{1}{4}n(n+1)-\frac{1}{2}\square_n$$

$$= \frac{1}{2}n\left(n+\frac{1}{2}\right)(n+1)-\frac{1}{2}\square_n$$

Review: eq. 2.33 (Upper Triangle Sum)

Consider the array of n^2 products $a_j a_k$. Our goal will be to find a simple formula for the sum of all elements on or above the diagonal of this array. Because $a_j a_k = a_k a_j$, the array is symmetrical about its main diagonal; therefore $S_\triangleleft$ will be approximately half the sum of all the elements.

$$\begin{pmatrix} a_1 a_1 & a_1 a_2 & \cdots & a_1 a_n \\ \vdots & & \ddots & \vdots \\ a_n a_1 & a_n a_2 & \cdots & a_n a_n \end{pmatrix}$$

Upper Triangle Sum (eq. 2.33) cont…

$$S_{\triangleleft} = \sum_{1\le j\le k\le n} a_j a_k = \sum_{1\le k\le j\le n} a_k a_j = \sum_{1\le k\le j\le n} a_j a_k = S_{\triangleleft}$$

because we can rename (j,k) as (k,j), Furthermore, since

$$\left(1\le j\le k\le n\right) \cap \left(1\le k\le j\le n\right) \equiv \left(1\le j,k\le n\right) \cap \left(1\le j=k\le n\right)$$

we have $2S_{\triangleleft} = S_{\triangleleft} + S_{\triangleleft} = \sum_{1\le j\le k\le n} a_j a_k + \sum_{1\le k\le j\le n} a_j a_k = \sum_{1\le j,k\le n} a_j a_k + \sum_{1\le j=k\le n} a_j a_k$

The first sum is $\left(\sum_{j=1}^{n} a_j\right)\left(\sum_{k=1}^{n} a_k\right) = \left(\sum_{k=1}^{n} a_k\right)^2$, by the general distributive law (2.28).

The second sum is $\left(\sum_{k=1}^{n} a_k^{\ 2}\right)$.

Therefore we have $S_{\triangleleft} = \sum_{1\le j\le k\le n} a_j a_k = \frac{1}{2}\left(\left(\sum_{k=1}^{n} a_k\right)^2 + \left(\sum_{k=1}^{n} a_k^{\ 2}\right)\right)$ (Equation 2.33)

An expression for the upper traingular sum in terms of simpler single sums.

Let's try to solve…

$$\blacksquare_n + \square_n = \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^2$$

$$= \sum_{k=1}^{n} \left(k^3 + k^2\right) \qquad \left[\text{associative law } (2.16)\right]$$

$$= \sum_{k=1}^{n} k^2 \left(k+1\right)$$

$$= \sum_{k=1}^{n} k * k * \left(k+1\right)$$

$$= \sum_{k=1}^{n} 2 * \frac{1}{2} * k * k * (k+1)$$

solution continues…

$$= 2\sum_{k=1}^{n} k * \frac{1}{2} * k * (k+1)$$

[distributive law (2.15)]

Now, $\frac{1}{2} * k * (k+1) = \sum_{j=1}^{k} j$ (sum of first k natural numbers)

$$2\sum_{k=1}^{n} k * \frac{1}{2} * k * (k+2) = 2\left(\sum_{k=1}^{n} k\right)\left(\sum_{j=1}^{k} j\right)$$

Now, we would use general distributive law (eq 2.28) that states

$$\sum_{\substack{j \in J \\ k \in K}} a_j b_k = \left(\sum_{j \in J} a_j\right)\left(\sum_{k \in K} b_k\right)$$

solution continues…

$$\sum_{k=1}^{n} k \sum_{j=1}^{k} j = \sum_{k=1}^{n}\sum_{j=1}^{k} k * j = \sum_{j=1}^{k}\sum_{k=1}^{n} k * j \qquad (as \text{ k is a constant with respect to j})$$

Since, $(1 \le k \le n) \cap (1 \le j \le k) \equiv (1 \le j \le k \le n)$

Therefore, $\sum_{j=1}^{k} \sum_{k=1}^{n} k * j = \sum_{1 \le j \le k \le n} k * j$

So, our original equation becomes –

$$2 \sum_{k=1}^{n} k * \frac{1}{2} * k * (k+2) = 2\left(\sum_{k=1}^{n} k\right)\left(\sum_{j=1}^{k} j\right)$$

$$= 2 \sum_{1 \le j \le k \le n} k * j$$

We Proved

$$\blacksquare_n + \square_n = 2 \sum_{1 \le j \le k \le n} jk$$

Use equation (2.33)

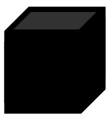
We use the expression for upper triangular sum (equation 2.33) for further evaluation as shown below:-

$$\blacksquare_n + \square_n = 2 * \frac{1}{2}\left(\left(\sum_{k=1}^{n} k\right)^2 + \sum_{k=1}^{n} k^2\right)$$

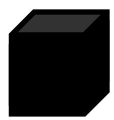
$$= \left(\sum_{k=1}^{n} k\right)^2 + \sum_{k=1}^{n} k^2$$

The first term is the square of the summation of the first n natural numbers and the second term is the sum of the first n squares, i.e., $\square_n$

Finally the solution ☺

 $_n$ + $\square_n$ = $\left(\frac{n(n+1)}{2}\right)^2 + \square_n$

Therefore,

 $_n$ = $\left(\frac{n(n+1)}{2}\right)^2$

Verification of solution

We use the same tabular method to prove the accuracy of our solution

n	1	2	3	4	5	6	7	8	9	10	…
N^3	**1**	**8**	**27**	**64**	**125**	**216**	**343**	**512**	**729**	**1000**	**…**
■ n	**1**	**9**	**36**	**100**	**225**	**441**	**784**	**1296**	**2025**	**3025**	**…**
$\left[\frac{n(n+1)}{2}\right]^2$	**1**	**9**	**36**	**100**	**225**	**441**	**784**	**1296**	**2025**	**3025**	**…**

Intuitive Proof

$1^3 = 1^2$

$1^3 + 2^3 = 9 = (1+2)^2$

$1^3 + 2^3 + 3^3 = 36 = (1+2+3)^2$

$1^3 + 2^3 + 3^3 + 4^3 = 100 = (1+2+3+4)^2$

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$1^3 + 2^3 + 3^3 + \ldots + n^3 = (1 + 2 + 3 + \ldots + n)^2$

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k \right)^2 = \left[\frac{n(n+1)}{2} \right]^2$$