cse547
Problems 5,7
Chapter 2

Problem 5

- Question: What's wrong with the following derivation?

$$\left(\sum_{j=1}^{n} a_j\right)\left(\sum_{j=1}^{n} \frac{1}{a_k}\right) = \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{a_j}{a_k} = \sum_{k=1}^{n}\sum_{k=1}^{n} \frac{a_k}{a_k} = \sum_{k=1}^{n}\sum_{k=1}^{n} n = n^2$$

Correctness verification (1)

- We want to see whether the derivation is correct or not
- For this we set $n = 3$ and we want to see if the right part of the derivation is equal to the left part

Correctness verification (2)

$$S_L = \left(\sum_{j=1}^{3} a_j\right)\left(\sum_{j=1}^{3} \frac{1}{a_k}\right)$$

$$S_L = \left(\sum_{j=1}^{3} a_j\right)\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}\right)$$

$$S_L = (a_1 + a_2 + a_3)\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}\right)$$

$$S_L = a_1\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}\right) + a_2\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}\right) + a_3\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}\right)$$

$$S_L = 1 + \frac{a_1}{a_2} + \frac{a_1}{a_3} + \frac{a_2}{a_1} + 1 + \frac{a_2}{a_3} + a_3\left(\frac{a_3}{a_1} + \frac{a_3}{a_2} + 1\right)$$

$$S_L = 3 + \frac{a_2 + a_3}{a_1} + \frac{a_1 + a_3}{a_2} + \frac{a_1 + a_2}{a_3}$$

Correctness verification (3)

$$S_R = \sum_{j=1}^{3}\sum_{k=1}^{3}\frac{a_k}{a_k}$$

$$S_R = \sum_{j=1}^{3}\left(\frac{a_1}{a_1} + \frac{a_2}{a_2} + \frac{a_3}{a_3}\right)$$

$$S_R = \sum_{j=1}^{3} 3$$

$$S_R = 9 = 3^2$$

We can see that S_L != S_R, so we detect that the derivation is not correct

Correctness – General case

- We could have inferred that the derivation is not correct also if we try to express S_L and S_R in the general case!
- By eliminating the sum sign (as we did in the previous example and also see pages 49, 90 –> “General distributive law” in the notes and in the book too) we get:

$$S_L = n + \frac{a_2 + a_3 + ... + a_n}{a_1} + \frac{a_1 + a_3 + a_4 + ... + a_n}{a_2} +$$

$$+ ... + \frac{a_1 + ... + a_{k-1} + a_k + ... + a_n}{a_k} + ... + \frac{a_1 + a_2 + ... + a_{n-1}}{a_n}$$

$$S_R = n^2, \forall a_j\, j = 1..n, \forall a_k k = 1..n$$

$$\Rightarrow S_R \neq S_L$$

How can we find the error?

- Idea: check every step of the derivation
- We have 2 derivation steps (see below):

$$\left(\sum_{j=1}^{n} a_j\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right) \overset{1}{=} \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{a_j}{a_k} \overset{2}{=} \sum_{k=1}^{n}\sum_{k=1}^{n} \frac{a_k}{a_k} = \sum_{k=1}^{n}\sum_{k=1}^{n} n = n^2$$

- We want to check which one is wrong

Derivation step 1

- We check the first step of the derivation

$$\left(\sum_{j=1}^{n} a_j\right)\left(\sum_{j=1}^{n} \frac{1}{a_k}\right) \overset{1}{=} \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{a_j}{a_k}$$

- Can we do this step? Yes
- Why? Based on the "General Distributive Law" that we proved in class (see pages 90,91 in the lecture notes)

Derivation step 2

- We check the first step of the derivation

$$\left(\sum_{j=1}^{n} a_j\right)\left(\sum_{j=1}^{n} \frac{1}{a_k}\right) \overset{1}{=} \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{a_j}{a_k}$$

- Can we do this step? Yes
- Why? Based on the “General Distributive Law” that we proved in class (see pages 90-99 in the lecture notes)

Derivation step 2

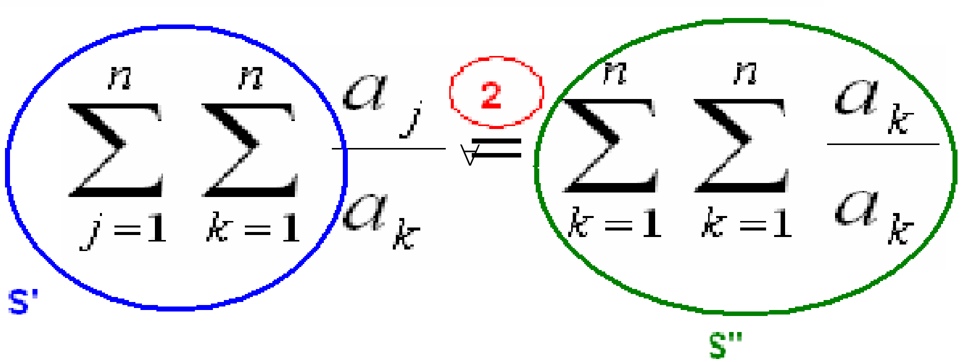
- We check the first step of the derivation

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \frac{a_j}{a_k} \overset{2}{=} \sum_{k=1}^{n} \sum_{k=1}^{n} \frac{a_k}{a_k}$$

- Can we do this step? No!
- Why? Because the transformation is not in accordance with the changing of the indexes in multiple sums rule.

Derivation step 2

- We check the first step of the derivation
- From logic we know that in the multiple sum S' in this step, k is a bound variable to the inner sum, while j is a bound variable to the exterior sum.



$$\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{a_j}{a_k} = \sum_{k=1}^{n}\sum_{k=1}^{n}\frac{a_k}{a_k}$$

- But in the multiple sum S'', k is a bound variable both to the inner sum, and to the exterior sum.
- Based on the substitution rules of predicate logic, we cannot substitute j of the outer sum with the same k as the one in the inner sum.

Derivation 2

- The substitution works only when $\mathrm{a}_{\mathrm{j}} = \mathrm{a}_{\mathrm{k}}, \forall i, j, 1 \le j, k \le n.$
- Why? Because then we will have:

$$S_L = n + \frac{a_2 + a_3 + \ldots + a_n}{a_1} + \frac{a_1 + a_3 + a_4 + \ldots + a_n}{a_2} +$$

$$+ \ldots + \frac{a_1 + \ldots + a_{k-1} + a_k + \ldots + a_n}{a_k} + \ldots + \frac{a_1 + a_2 + \ldots + a_{n-1}}{a_n}$$

$$S_L = n + \frac{(n-1)a_1}{a_1} + \ldots + \frac{(n-1)a_n}{a_n}$$

$$S_L = n + (n-1)n = n^2 = S_R$$

Problem 7

- Let $\nabla f(x) = f(x) - f(x-1)$. What is $\nabla\left(x^{\overline{m}}\right)$?

- We define the Rising Factorial Power, $x^{\overline{m}}$, (see page 48 in the book) as:

$$x^{\overline{m}} = x(x+1)...(x+m-1), m > 0$$

- We want to evaluate:

$$\nabla\left(x^{\overline{m}}\right) = x^{\overline{m}} - (x-1)^{\overline{m}}$$

Problem 7

- We define the Rising Factorial Power, $\left(x^{\overline{m}}\right)$, (see page 144 of the notes or book page 48) as:

$$x^{\overline{m}} = x(x+1)\ldots(x+m-1), m > 0$$

- We want to evaluate:

$$\nabla\left(x^{\overline{m}}\right) = x^{\overline{m}} - (x-1)^{\overline{m}}$$

- To prove:

$$\nabla\left(x^{\overline{m}}\right) = mx^{\overline{m-1}}$$

Problem 7

- In other words we want to say that the ∇ operator behaves like the D operator defined in the book (see page 47) the same as that for the notes (page 139)

- In order to prove this we first evaluate:

$$x^{\overline{m}} = x(x+1)(x+2)\ldots(x+m-1)$$

- Then we evaluate:

$$(x-1)^{\overline{m}} = (x-1)x(x+1)(x+2)\ldots(x-1+m-1)$$

$$= (x-1)x(x+1)(x+2)\ldots(x+m-2)$$

Problem 7

- Now we can evaluate:

$$\nabla\left(x^{\overline{m}}\right) = x^{\overline{m}} - (x-1)^{\overline{m}}$$
$$= x(x+1)(x+2)\ldots(x+m-1) - (x-1)x(x+1)(x+2)\ldots(x+m-2)$$
$$= x(x+1)(x+2)\ldots(x+m-2)(x+m-1-x+1)$$
$$= x(x+1)(x+2)\ldots(x+m-2)m$$
$$= mx^{\overline{m-1}}, q.e.d$$

- Final result:

$$\nabla\left(x^{\overline{m}}\right) = mx^{\overline{m-1}}$$

Problem 7

- In the book the definition for the $\nabla\left(x^{\overline{m}}\right)$ operator is:

$$\nabla\left(x^{\overline{m}}\right) = x^{\overline{m}} - \left(x-1\right)^{\overline{m}}$$

- We want to see what is the result if we use the definition for the Δ operator (see class notes definition page 144 and book page 47):

$$\Delta f(x) = f(x+1) - f\left(x\right)$$

Problem 7

- We first evaluate:

$$(x+1)^{\overline{m}} = (x+1)(x+2)\ldots(x+1+m-1)$$
$$= (x+1)(x+2)\ldots(x+m)$$

- Then we evaluate:

$$x^{\overline{m}} = x(x+1)(x+2)\ldots(x+m-1)$$

- In the end we get

$$\boxed{\Delta\left(x^{\overline{m}}\right)} = (x+1)^{\overline{m}} - x^{\overline{m}}$$
$$= (x+1)(x+2)\ldots(x+m) - x(x+1)(x+2)\ldots(x+m-1)$$
$$= (x+1)(x+2)\ldots(x+m-1)(x+m-x)$$
$$= (x+1)(x+2)\ldots(x+m-1)m$$
$$\boxed{= m(x+1)^{\overline{m-1}}}$$

This is the result obtained using the Δ formula!

Problem 7

- We obtained:

$$\Delta\left(x^{\overline{m}}\right) = m(x+1)^{\overline{m-1}} \neq mx^{\overline{m-1}} = \nabla\left(x^{\overline{m}}\right)$$

This is the result obtained using the Δ formula!

This is the book result using the ∇ operator!

Thank you!

Questions?