CHAPTER 2

INFINITE SUMS (SERIES)

Lecture Notes PART 1

We extend now the notion of a finite sum $\Sigma_{k=1}^{n}\ a_k$ to an INFINITE SUM which we write as

$$\Sigma_{n=1}^{\infty}\ a_n$$

as follows.

For a given a sequence $\{a_n\}_{n\in N-\{0\}}$, i.e the sequence

$$a_1, a_2, a_3, ...a_n, ...$$

we consider a following (infinite) sequence

$$S_1 = a_1, \quad S_2 = a_1 + a_2, \quad ...., \quad S_n = \Sigma_{k=1}^{n}\ a_k, \quad S_{n+1} = \Sigma_{k=1}^{n+1}\ a_k, \quad ....$$

and define the infinite sum as follows.

DEFINITION 1
If the limit of the sequence $\{S_n = \Sigma_{k=1}^{n}\ a_k\}$ exists we call it an INFINITE SUM of the sequence S_n and write it as

$$\Sigma_{n=1}^{\infty}\ a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \Sigma_{k=1}^{n}\ a_k.$$

The sequence $\{S_n\}$ is called its sequence of partial sums.

DEFINITION 2
If the limit $\lim_{n\to\infty} S_n$ exists and is finite, i.e.

$$\lim_{n\to\infty} S_n = S,$$

then we say that the infinite sum $\Sigma_{n=1}^{\infty}\ a_n$ CONVERGES to S and we write

$$\Sigma_{n=1}^{\infty}\ a_n = \lim_{n\to\infty} \Sigma_{k=1}^{n}\ a_k = S,$$

otherwise the infinite sum DIVERGES.

In a case that $\lim_{n\to\infty} S_n$ exists and is infinite, i.e. $\lim_{n\to\infty} S_n = \infty$, then we say that the infinite sum $\Sigma_{n=1}^{\infty}\ a_n$ DIVERGES to ∞ and we write

$$\Sigma_{n=1}^{\infty}\ a_n = \infty.$$

In a case that $\lim_{n\to\infty} S_n$ does not exist we say that the infinite sum $\Sigma_{n=1}^{\infty}\ a_n$ DIVERGES.

OBSERVATION 1
In a case when all elements of the sequence $\{a_n\}_{n\in N-\{0\}}$ are equal 0 starting from a certain $k \geq 1$ the infinite sum becomes a finite sum, hence the infinite sum is a generalization of the finite one, and this is why we keep the similar notation.

EXAMPLE 1
The infinite sum of a geometric sequence $a_n = x^k$ for $x \geq 0$, i.e. $\Sigma_{n=1}^{\infty}\ x^n$ converges if and only if $\mid x \mid < 1$ because

$$\Sigma_{k=1}^{n}\ x^k = S_n = \frac{x - x^{n+1}}{x-1}, \quad \text{and}$$

$$\lim_{n\to\infty} \frac{x(1-x^n)}{x-1} = \lim_{n\to\infty} \frac{x}{x-1}(1-x^n) = \frac{x}{x-1} \quad \text{iff} \quad \mid x \mid < 1,$$

hence

$$\Sigma_{n=1}^{\infty} x^k = \frac{x}{x-1}.$$

EXAMPLE 2
The series $\Sigma_{n=1}^{\infty}\ 1$ DIVERGES to ∞ as $S_n = \Sigma_{k=1}^{n} 1 = n$ and $\lim_{n\to\infty} S_n = \lim_{n\to\infty} n = \infty$.

EXAMPLE 3
The infinite sum $\Sigma_{n=1}^{\infty}\ (-1)^n$ DIVERGES.

EXAMPLE 4
The infinite sum $\Sigma_{n=0}^{\infty} \frac{1}{(k+1)(k+2)}$ CONVERGES to 1; i.e.

$$\Sigma_{n=0}^{\infty} \frac{1}{(k+1)(k+2)} = 1.$$

Proof: first we evaluate $S_n = \Sigma_{k=1}^{n} \frac{1}{(k+1)(k+2)}$ as follows.

$$S_n = \Sigma_{k=0}^{n} \frac{1}{(k+1)(k+2)} = \Sigma_{k=1}^{n} \underline{k^{-2}} = -\frac{1}{x+1}\Big|_0^{n+1} = -\frac{1}{n+2} + 1 \text{ and}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} -\frac{1}{n+2} + 1 = 1.$$

DEFINITION 3
For any infinite sum (series)

$$\Sigma_{n=1}^{\infty} a_n$$

a series

$$r_n = \Sigma_{m=n+1}^{\infty}\ a_m$$

is called its n-th REMINDER.

FACT 1
If $\Sigma_{n=1}^{\infty} a_n$ converges, then so does its n-th REMINDER $r_n = \Sigma_{m=n+1}^{\infty}\ a_m$.

Proof: first, observe that if $\Sigma_{n=1}^{\infty} a_n$ converges, then for any value on n so does $r_n = \Sigma_{m=n+1}^{\infty}\ a_m$ because

$$r_n = \lim_{n\to\infty}(a_{n+1} + ... + a_{n+k}) = \lim_{n\to\infty} S_{n+k} - S_n = \Sigma_{m=1}^{\infty} a_m\ \ - S_n.$$

So we get

$$\lim_{n\to\infty}\ r_n = \Sigma_{m=1}^{\infty} a_m\ \ - \lim_{n\to\infty}\ S_n = \Sigma_{m=1}^{\infty} S_m\ \ -\ \Sigma_{n=1}^{\infty} a_n = S - S = 0.$$

General Properties of Infinite Sums

THEOREM 1
If the infinite sum

$$\Sigma_{n=1}^{\infty} a_n \text{ converges, then } \lim_{n\to\infty}\ a_n = 0.$$

Proof: observe that $a_n = S_n - S_{n-1}$ and hence

$$\lim_{n\to\infty}\ a_n = \lim_{n\to\infty}\ S_n\ The - \lim_{n\to\infty}\ S_{n-1} = 0,$$

as $\lim_{n\to\infty}\ S_n = \lim_{n\to\infty}\ S_{n-1}$.

REMARK 1
The reverse statement to the theorem 1

$$\text{If } \lim_{n\to\infty} a_n = 0. \text{ then } \Sigma_{n=1}^{\infty} a_n \text{ converges}$$

is **not always true**. There are infinite sums with the term converging to zero that are not convergent.

EXAMPLE 5
The infinite HARMONIC sum

$$H = \Sigma_{n=1}^{\infty} \frac{1}{n}$$

DIVERGES to ∞, i.e. $\Sigma_{n=1}^{\infty} \frac{1}{n} = \infty$ but $\lim_{n\to\infty} \frac{1}{n} = 0$.

DEFINITION 4
Infinite sum $\Sigma_{n=1}^{\infty} a_n$ is BOUNDED if its sequence of partial sums $S_n = \Sigma_{k=1}^{n} a_k$ is BOUNDED; i.e. there is a number M such that $|S_n| < M$, for all $n \leq 1, n \in N$.

FACT 2
Every convergent infinite sum is bounded.

THEOREM 2
If the infinite sums $\Sigma_{n=1}^{\infty} a_n$, $\Sigma_{n=1}^{\infty} b_n$ CONVERGE, then the following properties hold.

$$\Sigma_{n=1}^{\infty}(a_n + b_n) = \Sigma_{n=1}^{\infty} a_n \; + \; \Sigma_{n=1}^{\infty} b_n,$$

$$\Sigma_{n=1}^{\infty} c a_n = c \Sigma_{n=1}^{\infty} a_n, \; c \in R.$$

Alternating Infinite Sums. Abel Theorem

DEFINITION 5
An infinite sum

$$\Sigma_{n=1}^{\infty} (-1)^{n+1} a_n, \quad \text{for} \quad a_n \geq 0$$

is called ALTERNATING infinite sum (alternating series).

EXAMPLE 6
Consider

$$\Sigma_{n=1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 - 1 + ...$$

If we group the terms in pairs, we get

$$(1-1) + (1-1) + ... = 0$$

but if we start the pairing one step later, we get

$$1 - (1-1) - (1-1) - ..... = 1 - 0 - 0 - 0 - ... = 1.$$

It shows that grouping terms in a case of infinite sum can lead to inconsistencies (contrary to the finite case). Look also example on page 59. We need to develop some strict criteria for manipulations and convergence/divergence of alternating series.

THEOREM 3
The alternating infinite sum $\Sigma_{n=1}^{\infty}(-1)^{n+1}a_n, (a_n \geq 0)$ such that

$$a_1 \geq a_2 \geq a_3 \geq .... \text{ and } \lim_{n\to\infty} a_n = 0$$

always CONVERGES. Moreover, its partial sums $S_n = \Sigma_{k=1}^{n}(-1)^{n+1}a_n$ fulfil the condition

$$S_{2n} \leq \Sigma_{n=1}^{\infty}(-1)^{n+1}a_n \leq S_{2n+1}.$$

Proof: observe that the sequence of S_{2n} is increasing as

$$S_{2(n+1)} = S_{2n+2} = S_{2n}+(a_{2n+1}-a_{2n+2}, \text{ and } a_{2n+1}-a_{2n+2} \geq 0, i.e. S_{2n+2} \geq S_{2n}.$$

The sequence of S_{2n} is also bounded as

$$S_{2n} = a_1 - ((a_2 - a_3) + (a_4 - a_5) + ...a_{2n}) \leq a_1.$$

We know that any bounded and increasing sequence is is convergent, so we proved that S_{2n} converges. Let denote $\lim_{n\to\infty} S_{2n} = g$.
To prove that $\Sigma_{n=1}^{\infty}(-1)^{n+1}a_n = \lim_{n\to\infty} S_n$ converges we have to show now that $\lim_{n\to\infty} S_{2n+1} = g$.
Observe that $S_{2n+1} = S_{2n} + a_{2n+2}$ and we get

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} a_{2n+2} = g$$

as we assumed that $\lim_{n\to\infty} a_n = 0$.
We proved that the sequence $\{S_{2n}\}$ is increasing, in a similar way we prove that the sequence $\{S_{2n+1}\}$ is decreasing. Hence $S_{2n} \leq \lim_{n\to\infty} S_{2n} = g = \Sigma_{n=1}^{\infty}(-1)^{n+1}a_n$ and $S_{2n+1} \geq \lim_{n\to\infty} S_{2n+1} = g = \Sigma_{n=1}^{\infty}(-1)^{n+1}a_n$, i.e

$$S_{2n} \leq \Sigma_{n=1}^{\infty}(-1)^{n+1}a_n \leq S_{2n+1}.$$

EXAMPLE 7
Consider the INHARMONIC series (infinite sum)

$$AH = \Sigma_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}.....$$

Observe that $a_n = \frac{1}{n}$, and $\frac{1}{n} \geq \frac{1}{n+1}$ i.e. $a_n \geq a_{n+1}$ for all n, so the assumptions of the theorem 3 are fulfilled for AH and hence AH **converges**. In fact, it is proved (by analytical methods, not ours) that

$$AH = \Sigma_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = \ln 2.$$

EXAMPLE 8
A series (infinite sum)

$$\Sigma_{n=0}^{\infty}(-1)^n \frac{1}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}......$$

converges, by **Theorem 3** (proof similar to the one in the example 7). It also is proved that

$$\Sigma_{n=0}^{\infty}(-1)^n \frac{1}{2n+1} = \frac{\pi}{4}.$$

THEOREM 4 (ABEL Theorem)
IF a sequence $\{a_n\}$ fulfils the assumptions of the **theorem 3** i.e.

$$a_1 \geq a_2 \geq a_3 \geq .... \text{ and } \lim_{n\to\infty} a_n = 0$$

and an infinite sum (converging or diverging) $\Sigma_{n=1}^{\infty} b_n$ is bounded,
THEN the infinite sum

$$\Sigma_{n=1}^{\infty} a_n b_n$$

always **converges**.

Observe that Theorem 3 is a special case of theorem 4 when $b_n = (-1)^{n+1}$.

Convergence of Infinite Sums with Positive Terms

We consider now infinite sums with all its terms being positive real numbers, i.e.

$$S = \Sigma_{n=1}^{\infty} a_n, \text{ for } a_n \geq 0, a_n \in R.$$

Observe that if all $a_n \geq 0$, then the sequence $\{S_n\}$ of partial sums $S_n = \Sigma_{k=1}^{n} a_k$ is increasing; i.e.

$$S_1 \leq S_2 \leq .... \leq S_n...$$

and hence the $\lim_{n\to\infty} S_n$ exists and is finite or is ∞. This proves the following theorem.

THEOREM 5
The infinite sum

$$S = \Sigma_{n=1}^{\infty} a_n, \text{ for } a_n \geq 0, a_n \in R$$

always **converges**, or **diverges** to ∞.

THEOREM 6 (Comparing the series)
Let $\Sigma_{n=1}^{\infty} a_n$ be an infinite sum and $\{b_n\}$ be a sequence such that for all n,

$$0 \leq b_n \leq a_n.$$

If the infinite sum $\Sigma_{n=1}^{\infty} a_n$ **converges**, then $\Sigma_{n=1}^{\infty} b_n$ also **converges** and

$$\Sigma_{n=1}^{\infty} b_n \ \leq \ \Sigma_{n=1}^{\infty} a_n.$$

Proof: Denote

$$S_n = \Sigma_{k=1}^{n} a_k, \quad T_n = \Sigma_{k=1}^{n} b_k.$$

As $0 \leq b_n \leq a_n$ we get that also $S_n \leq T_n$. But

$$S_n \leq \lim_{n \to \infty} S_n = \Sigma_{n=1}^{\infty} a_n \ \text{ so also } \ T_n \leq \Sigma_{n=1}^{\infty} a_n = S.$$

The inequality $T_n \leq \Sigma_{n=1}^{\infty} a_n = S$ means that the sequence $\{T_n\}$ is a bounded sequence with positive terms, hence by **theorem 5**, it converges.

By the assumption that $\Sigma_{n=1}^{\infty} a_n$ we get that

$$\Sigma_{n=1}^{\infty} a_n = \lim_{n \to \infty} \Sigma_{k=1}^{n} a_k = \lim_{n \to \infty} S_n = S.$$

We just proved that $T_n = \Sigma_{k=1}^{n} b_k$ converges, i.e.

$$\lim_{n \to \infty} T_n = T = \Sigma_{n=1}^{\infty} b_n.$$

But also we proved that $S_n \leq T_n$, hence

$$\lim_{n \to \infty} S_n \leq \lim_{n \to \infty} T_n$$

what means that

$$\Sigma_{n=1}^{\infty} b_n \ \leq \ \Sigma_{n=1}^{\infty} a_n.$$

EXAMPLE 9
Let's use **Theorem 5** to prove that the series

$$\Sigma_{n=1}^{\infty} \frac{1}{(n+1)^2}$$

converges. We prove by analytical methods that it converges to $\frac{\pi^2}{6}$, here we prove only that it does converge. First observe that the series below converges to 1, i.e.

$$\Sigma_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

Consider

$$S_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} .... + \frac{1}{n(n+1)} = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + ...(\frac{1}{n} - \frac{1}{n+1}) = 1 - \frac{1}{n+1}$$

so we get

$$\Sigma_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n \to \infty} S_n = \lim_{n \to \infty} (1 - \frac{1}{n+1}) = 1.$$

Now we observe (easy to prove) that

$$\frac{1}{2^2} \leq \frac{1}{1 \cdot 2}, \ \frac{1}{3^2} \leq \frac{1}{1 \cdot 3}, \ ..... \ \frac{1}{(n+1)^2} \leq \frac{1}{n(n+1)}, ...$$

i.e. we proved that all assumptions of **Theorem 5** hold, hence $\Sigma_{n=1}^{\infty} \frac{1}{(n+1)^2}$ converges and moreover

$$\Sigma_{n=1}^{\infty} \frac{1}{(n+1)^2} \ \leq \ \Sigma_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

THEOREM 7 (D'Alambert's Criterium)
Any series with all its terms being positive real numbers, i.e.

$$\Sigma_{n=1}^{\infty} a_n, \text{ for } a_n \geq 0, a_n \in R$$

converges if the following condition holds:

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} \ < 1.$$

Proof: let h be any number such that $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} \ < h < 1$. It means that there is k such that for any $n \geq k$ we have $\frac{a_{n+1}}{a_n} < h$, i.e. $a_{n+1} < h a_n$ and

$$a_{k+1} < a_k h, \ \ a_{k+2} = a_{k+1} h < a_k h^2, \ \ a_{k+3} = a_{k+2} h < a_k h^3, ...$$

i.e. all terms a_n of $\Sigma_{n=k}^{\infty} a_n$ are smaller then the terms of a converging (as $0 < h < 1$) geometric series $\Sigma_{n=0}^{\infty} a_k h^n = a_k + a_k h + a_k h^2 + ...$ By **Theorem 5** the series $\Sigma_{n=1}^{\infty} a_n$ must converge.

THEOREM 8 (Cauchy's Criterium)
Any series with all its terms being positive real numbers, i.e.

$$\Sigma_{n=1}^{\infty} a_n, \text{ for } a_n \geq 0, a_n \in R$$

converges if the following condition holds:

$$\lim_{n \to \infty} \sqrt[n]{a_n} \ < 1.$$

Proof: we carry the proof in a similar way as the proof of theorem 6. Let h be any number such that $\lim_{n\to\infty} \sqrt[n]{a_n} < h < 1$. So it means that there is k such that for any $n \geq k$ we have $\sqrt[n]{a_n} < h$, i.e. $a_n < h^n$. This means that all terms a_n of $\Sigma_{n=k}^{\infty} a_n$ are smaller then the terms of a converging (as $0 < h < 1$) geometric series $\Sigma_{n=k}^{\infty} h^n = h^k + h^{k+1} + h^{k+2} + \ldots$. By **Theorem 5** the series $\Sigma_{n=1}^{\infty} a_n$ must converge.

THEOREM 9 (Divergence Criteria
Any series with all its terms being positive real numbers, i.e.

$$\Sigma_{n=1}^{\infty} a_n, \text{ for } a_n \geq 0, a_n \in R$$

diverges if

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1 \quad \text{or} \quad \lim_{n\to\infty} \sqrt[n]{a_n} > 1$$

Proof: observe that if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$, then for sufficiently large n we have that

$$\frac{a_{n+1}}{a_n} > 1, \quad \text{and hence} \quad a_{n+1} > a_n.$$

This means that the limit of the sequence $\{a_n\}$ can't be 0. By **theorem 1** we get that $\Sigma_{n=1}^{\infty} a_n$ diverges.
Similarly, if $\lim_{n\to\infty} \sqrt[n]{a_n} > 1$, then then for sufficiently large n we have that

$$\sqrt[n]{a_n} > 1 \quad \text{and hence} \quad a_n > 1,$$

what means that the limit of the sequence $\{a_n\}$ can't be 0. By **theorem 1** we get that $\Sigma_{n=1}^{\infty} a_n$ diverges.

REMARK 2
It can happen that for a certain infinite sum $\Sigma_{n=1}^{\infty} a_n$)

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 = \lim_{n\to\infty} \sqrt[n]{a_n}.$$

In this case our Divergence Criteria do not decide whether the infinite sum converges or diverges. In this case we say that the infinite sum **DOES NOT REACT** on the criteria.

EXAMPLE 10
The Harmonic series

$$H = \Sigma_{n=1}^{\infty} \frac{1}{n}$$

does not react on **D'Alambert's Criterium** (Theorem 7) because

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})} = 1.$$

EXAMPLE 11
The series from example 9

$$\Sigma_{n=1}^{\infty} \frac{1}{(n+1)^2}$$

does not react on **D'Alambert's Criterium** (Theorem 7) because

$$\lim_{n\to\infty} \frac{(n+1)^2}{(n+2)^2} = \lim_{n\to\infty} \frac{n^2+2n+1}{n^2+4n+1} = \lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{1}{n^2}}{1+\frac{4}{n}+\frac{4}{n^2}} = 1.$$

REMARK 3 Both series $\Sigma_{n=1}^{\infty} \frac{1}{n}$ and $\Sigma_{n=1}^{\infty} \frac{1}{(n+1)^2}$ do not react on D'Alambert's, but first in divergent and the second is convergent.

There are more criteria for convergence, most known are Kumer's criterium and Raabe criterium.

EXAMPLE 12
The series

$$\Sigma_{n=1}^{\infty} \frac{c^n}{n!}$$

converges for $c > 0$.

Proof: Use D'Alambert Criterium.

$$\frac{a_{n+1}}{a_n} = \frac{c^{n+1}}{c^n} \cdot \frac{n!}{(n+1)!} = \frac{c}{n+1}$$

and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{c}{n+1} = 0 < 1.$$

EXAMPLE 13
The series

$$\Sigma_{n=1}^{\infty} \frac{n!}{n^n}$$

converges.

Proof: Use D'Alambert Criterium.

$$\frac{a_{n+1}}{a_n} = \frac{n!(n+1)}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = (n+1) \frac{n^n}{(n+1)^{n+1}} = \frac{a_{n+1}}{a_n} = \frac{(n+1)n^n}{(n+1)^n(n+1)} = (\frac{n}{n+1})^n = \frac{1}{(1+\frac{1}{n})^n}$$