Problem 28

- <u>Problem Description</u>

Solve the recurrence

$$a_0 = 1$$

$$a_n = a_{n-1} + \left\lfloor \sqrt{a_{n-1}} \right\rfloor \quad \text{for all } n>0$$

$$a_0 = 1 \quad a_n = a_{n-1} + \lfloor \sqrt{a_{n-1}} \rfloor \quad \text{for all } n>0$$

- Problem Solution

Observation:

The square root and floor operations make the recurrence hard to solve. What if a_{n-1} is the square of some integer?

That will get rid of
the **SQUARE ROOT** and **FLOOR** operations**!**

$$a_0 = 1 \quad a_n = a_{n-1} + \left\lfloor \sqrt{a_{n-1}} \right\rfloor \quad \text{for all } n>0$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	…
a_n	1	2	3	4	6	8	10	13	16	20	24	28	33	38	44	50	57	64	72	80	…

$$a_0 = 1 \quad a_n = a_{n-1} + \left\lfloor \sqrt{a_{n-1}} \right\rfloor \quad \text{for all n>0}$$

- If $a_n = m^2$, we can get

$$a_{n+1} = m^2 + m$$
$$a_{n+2} = m^2 + m + m = m^2 + 2m$$
$$a_{n+3} = m^2 + 2m + m = (m+1)^2 + m - 1$$
$$a_{n+4} = m^2 + 3m + m + 1 = (m+1)^2 + 2m$$
$$a_{n+5} = (m+1)^2 + 2m + (m+1) = (m+2)^2 + m - 2$$
$$a_{n+6} = m^2 + 5m + 2 + (m+2) = (m+2)^2 + 2m$$
$$a_{n+7} = m^2 + 6m + 4 + (m+2) = (m+3)^2 + m - 3$$
$$a_{n+8} = m^2 + 7m + 6 + (m+3) = (m+3)^2 + 2m$$

$$a_n = m^2$$

$$a_{n+1} = m^2 + m \qquad a_{n+2} = m^2 + 2m$$

$$a_{n+3} = (m+1)^2 + m - 1 \qquad a_{n+4} = (m+1)^2 + 2m$$

$$a_{n+5} = (m+2)^2 + m - 2 \qquad a_{n+6} = (m+2)^2 + 2m$$

$$a_{n+7} = (m+3)^2 + m - 3 \qquad a_{n+8} = (m+3)^2 + 2m$$

Observation :

If $a_n = m^2$, then we have

$$a_{n+2k+1} = (m+k)^2 + m - k \qquad 0 \le k \le m$$

$$a_{n+2k+2} = (m+k)^2 + 2m \qquad 0 \le k \le m$$

$$a_0 = 1 \qquad a_n = a_{n-1} + \left\lfloor \sqrt{a_{n-1}} \right\rfloor \qquad \text{for all } n>0$$

Proof of $a_{n+2k+1} = (m+k)^2 + m - k \quad 0 \le k \le m$

Using **Mathematical Induction**

1) When j=0, we have

$$a_{n+2j+1} = a_{n+1} = a_n + \left\lfloor \sqrt{a_n} \right\rfloor = m^2 + m = (m+0)^2 + m - 0$$

The equation holds.

$a_0 = 1 \quad a_n = a_{n-1} + \left\lfloor \sqrt{a_{n-1}} \right\rfloor$ for all n>0

Proof of $a_{n+2k+1} = (m+k)^2 + m - k \quad 0 \le k \le m$

2) Assume the equation holds for all j where $0 \le j \le k < m$, we now prove the equation will hold for j = k+1

$$a_{n+2k+2} = a_{n+2k+1} + \left\lfloor \sqrt{a_{n+2k+1}} \right\rfloor$$

$$= (m+k)^2 + m - k + \left\lfloor \sqrt{(m+k)^2 + m - k} \right\rfloor$$

$$\because (m+k+1)^2 = (m+k)^2 + 2(m+k) + 1$$

$$= (m+k)^2 + m - k + 3k + m + 1$$

$$> (m+k)^2 + m - k > (m+k)^2 \qquad (0 \le j \le k < m)$$

$$\therefore \left\lfloor \sqrt{(m+k)^2 + m - k} \right\rfloor = m + k$$

$$a_0 = 1 \qquad a_n = a_{n-1} + \left\lfloor \sqrt{a_{n-1}} \right\rfloor \qquad \text{for all n>0}$$

Proof of $\quad a_{n+2k+1} = (m+k)^2 + m - k \qquad 0 \le k \le m$

$$\therefore a_{n+2k+2} = (m+k)^2 + m - k + m + k$$

$$= (m+k)^2 + 2m$$

SO $\quad a_{n+2(k+1)+1} = a_{n+2k+3}$

$$= a_{n+2k+2} + \left\lfloor \sqrt{a_{n+2k+2}} \right\rfloor$$

$$= (m+k)^2 + 2m + \left\lfloor \sqrt{(m+k)^2 + 2m} \right\rfloor$$

$$\because (m+k+1)^2 = (m+k)^2 + 2m + 2k + 1$$

$$> (m+k)^2 + 2m > (m+k)^2$$

$$\therefore \left\lfloor \sqrt{(m+k)^2 + 2m} \right\rfloor = m + k$$

$a_0 = 1 \qquad a_n = a_{n-1} + \left\lfloor \sqrt{a_{n-1}} \right\rfloor$ for all n>0

Proof of $a_{n+2k+1} = (m+k)^2 + m - k \qquad 0 \le k \le m$

$$\therefore a_{n+2(k+1)+1} = (m+k)^2 + 2m + m + k$$

$$= (m+k)^2 + 1 + 2m + 2k + m - k - 1$$

$$= (m+k+1)^2 + m - (k+1)$$

so we proved the equation also holds for j = k+1

Hence we proved the equation

$$a_{n+2k+1} = (m+k)^2 + m - k \qquad 0 \le k \le m$$

Similarly we can prove

$$a_{n+2k+2} = (m+k)^2 + 2m \qquad 0 \le k \le m$$

$$a_0 = 1 \qquad a_n = a_{n-1} + \left\lfloor \sqrt{a_{n-1}} \right\rfloor \qquad \text{for all n>0}$$

Observation revisited

when $a_n = m^2$

$$a_{n+2k+1} = (m+k)^2 + m - k \qquad 0 \le k \le m$$

$$a_{n+2k+2} = (m+k)^2 + 2m \qquad 0 \le k \le m$$

Note: when k=m, we have

$$a_{n+2m+1} = (m+m)^2 + m - m = (2m)^2$$

$$a_0 = 1 \quad a_n = a_{n-1} + \left\lfloor \sqrt{a_{n-1}} \right\rfloor \quad \text{for all n>0}$$

Given an arbitrary n , if we can find out the corresponding n_0 such that $a_{n_0} = m^2$, then we can calculate $a_n = a_{n_0 + 2k + 1}$ or $a_n = a_{n_0 + 2k + 2}$ using our observation.

<u>The question is:</u> Given n, how can we find n_0 ?

Let's calculate several a_n to see if we can find some pattern.

$$a_0 = 1 \quad a_n = a_{n-1} + \left\lfloor \sqrt{a_{n-1}} \right\rfloor \quad \text{for all } n>0$$

n_0	a_{n0}	m
0	1	1
3	4	2
8	16	4
17	64	8
34	256	16
…	…	…

$$a_0 = 1 \quad a_n = a_{n-1} + \left\lfloor \sqrt{a_{n-1}} \right\rfloor \quad \text{for all } n>0$$

We now need to do two things:

1. Find a closed form for the first column of the previous table

2. Given an arbitrary **n**, find the corresponding n0 that appears in the first column of the previous table

If these two things are done, we can find a closed form for our recurrence.

$$a_0 = 1 \qquad a_n = a_{n-1} + \left\lfloor \sqrt{a_{n-1}} \right\rfloor \quad \text{for all } n>0$$

n_0	a_{n0}	m
0	1	1
3	4	2
8	16	4
17	64	8
34	256	16
…	…	…

$$a_0 = 1 \qquad a_n = a_{n-1} + \left\lfloor \sqrt{a_{n-1}} \right\rfloor \qquad \text{for all n>0}$$

Let M_n be the number sequence corresponding to the third column of the table. It's easy to see and prove by mathematical induction that

$$M_n = 2^n \qquad n \geq 0$$

Note: when k=m, we have

$$a_{n+2m+1} = (m+m)^2 + m - m = (2m)^2$$

Let T_n be the number sequence corresponding to the first column of the table. We have

$$T_0 = 0$$

$$T_n = T_{n-1} + 2M_{n-1} + 1 \qquad n \geq 1$$

$a_0 = 1 \qquad a_n = a_{n-1} + \left\lfloor \sqrt{a_{n-1}} \right\rfloor$ for all n>0

$$T_n = T_{n-1} + 2M_{n-1} + 1 = T_{n-1} + 2^n + 1$$

$$= T_{n-2} + 2^{n-1} + 1 + 2^n + 1$$

$$= T_{n-3} + 2^{n-2} + 1 + 2^{n-1} + 1 + 2^n + 1$$

$$= T_{n-4} + \ldots\ldots$$

$$= n + \sum_{k=1}^{n} 2^k = 2^{n+1} + n - 2$$

(It can be proved by Mathematical Induction.)

We find the closed form of T_n!!!

$$a_0 = 1 \qquad a_n = a_{n-1} + \left\lfloor \sqrt{a_{n-1}} \right\rfloor \quad \text{for all } n>0$$

Given an arbitrary N, we can always find an integer l such that $N \in [2^{l+1} + l - 2, 2^{l+2} + l - 1)$

and the corresponding $n_0 = 2^{l+1} + l - 2$

and the corresponding $m = 2^l$

CASE 1: If N is odd, we can use

$$a_N = a_{n_0 + 2k + 1} = (m+k)^2 + m - k$$

$a_0 = 1 \quad a_n = a_{n-1} + \lfloor \sqrt{a_{n-1}} \rfloor$ for all n>0

$$a_N = a_{n_0+2k+1} = (m+k)^2 + m - k$$

$$= [2^l + \frac{N-(2^{l+1}+l-2)-1}{2}]^2 + 2^l - \frac{N-(2^{l+1}+l-2)-1}{2} \; *$$

$$= \ldots.$$

$$= (\frac{N-l}{2})^2 + 2^{l+1} - \frac{1}{4}$$

* N= n_0+2k+1 and $n_0 = 2^{l+1} + l - 2$

(We skip some details and focus on the final closed form of a_N)

$$a_0 = 1 \qquad a_n = a_{n-1} + \left\lfloor \sqrt{a_{n-1}} \right\rfloor \quad \text{for all n>0}$$

CASE 2: If N is even, similarly, we have

$$a_N = a_{n_0 + 2k + 2} = (m + k)^2 + 2m$$

$$= [2^l + \frac{N - (2^{l+1} + l - 2) - 2}{2}]^2 + 2^{l+1}$$

$$= [2^l + \frac{N - 2^{l+1} - l}{2}]^2 + 2^{l+1}$$

$$= (\frac{N - l}{2})^2 + 2^{l+1}$$

$$a_0 = 1 \qquad a_n = a_{n-1} + \left\lfloor \sqrt{a_{n-1}} \right\rfloor$$ for all n>0

Conclusion:

Given N (where $N \in [2^{l+1} + l - 2, 2^{l+2} + l - 1)$), we have

$$a_N = \left(\frac{N - l}{2}\right)^2 + 2^{l+1}$$ if N is even

$$a_N = \left(\frac{N - l}{2}\right)^2 + 2^{l+1} - \frac{1}{4}$$ if N is odd