CHAPTER 4 PROBLEM 45

The Number 9376 has a peculiar self-reproducing property that

$(9376)^2 = 97909376$

How many 4 Digit numbers x satisfy the equation

$x^2 \bmod 10000 = x$?

[No Hints in Text Book ☹]

So the Problem can be restated as

$x^2 \bmod 10^4 = x$

$x^2 \equiv x \pmod{10^4}$

[From textbook]

But we will prove it for general formula for n digits

$x^2 \equiv x \pmod{10^n}$ ……… (1)

$x \equiv x \pmod{10^n}$ ....(2)

[by Definition of Mod]

(1) – (2)

$$x^2 - x \equiv x - x \pmod{10^n}$$

$$x(x - 1) \equiv 0 \pmod{10^n}$$

we know that we can subtract congruence elements without losing congruence

Also, from

$x(x - 1) \equiv 0 \pmod{10^n}$,

we have

$x(x - 1) \equiv 0 \pmod{2^n}$

$x(x - 1) \equiv 0 \pmod{5^n}$

[By Theorem of Independent Residues]

$x \bmod 2^n$ = [0 or 1]
[either x or (x-1) have to be odd or even]

$x \bmod 5^n$ = [0 or 1]
[either x or (x-1) has to be a multiple of 5, x has to be 5 or 6]

$x = 0 \mid x = 1 \mid x = 5 \mid x = 6$

First two hold good only when

$n = 1$

First Solution:

$x \equiv 1 \pmod{2^n}$

$x \equiv 0 \pmod{5^n}$

Second Solution:

$x \equiv 0 \pmod{2^n}$

$x \equiv 1 \pmod{5^n}$

Sum of the two Solutions is

$10^n + 1$ (from Wiki)

Thus the solutions are
x and $10^n + 1 - x$

For $n = 4$
x and $10^4 + 1 - x$

we know x can be 9376
so the other number is
$10000 + 1 - 9376 = 625$
But this is not a 4 digit number.

Thus for n = 4 there is only one 4 digit Automorphic Number.

But in general, for each n, there are two n digit numbers [Not Proved]

References

www.Wikipedia.com
www.Mathword.com