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Chapter 5 Problem 14

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- *Prove identity (5.25) by negating the upper index in Vandermonde's convolution (5.22). Then show another negation yeilds (5.26).*

Useful Identities

$$\binom{n}{k} = \binom{n}{n-k}$$

Symmetry

$n \geq 0$

$n \in \mathbf{Z}$

$k \in \mathbf{Z}$

$$\binom{r}{k} = (-1)^k \binom{k-r-1}{k}$$

Upper Negation $k \in \mathbf{Z}$

$$\sum_k \binom{r}{m+k}\binom{s}{n-k} = \binom{r+s}{m+n}$$

Vandermonde's Convolution $n \in \mathbf{Z}$

Vandermonde's Convoluton states that the sum (over all integers k) of the product of two binomial coefficients, in which the upper indices are constant and lower indices have a constant sum for all k, is the binomial coefficient obtained by summing both lower and upper indices

What we need to prove

$$\sum_{k \le l} \binom{l-k}{m} \binom{s}{k-n} (-1)^k = (-1)^{l+m} \binom{s-m-1}{l-m-n}$$

$l, m, n \in \mathbf{Z}$;

$l, m, n \ge 0$

Identity 5.25

$$\sum_{0 \le k \le l} \binom{l-k}{m} \binom{q+k}{n} = \binom{l+q+1}{m+n+1}$$

$l, m \ge 0$;

$n \ge q \ge 0$

$l, m, n, q \in \mathbf{Z}$

Identity 5.26

Proving Identity 5.25

Starting with the left hand side of 5.25 we will derive the right hand side by upper negation and Vandermonde's Convolution. Beginning with the left hand side we have:

$$\sum_{k \le l} \binom{l-k}{m} \binom{s}{k-n} (-1)^k = \sum_{k \le l} \binom{l-k}{m} \binom{s}{k-n} (-1)^{k-m+l} (-1)^{m+l}$$

Since $(-1)^k = (-1)^k (1) = (-1)^k (-1)^{2l} = (-1)^{k+2l} = (-1)^{k+l+l-m+m} = (-1)^{k-m+l} (-1)^{m+l}$

Also since l and m are independent of k we have:

$$(-1)^{m+l} \sum_{k \le l} \binom{l-k}{m} \binom{s}{k-n} (-1)^{k-m+l}$$

Proving Identity 5.25 (cont.)

Now since $k \le l$, $l, m, n \in \mathbf{Z}$;, *and* $l, m, n \ge 0$ *we can use our symmetry property on the left hand side to obtain:*

$$(-1)^{l+m} \sum_{k \le l} \binom{l-k}{l-k-m} \binom{s}{k-n} (-1)^{l-k-m}$$

Since we have the same exponent of -1 as the bottom part of the first term we can use upper negation once we get the upper part of the first term in the same form.

$$(-1)^{l+m} \sum_{k \le l} \binom{l-k+m-m+1-1}{l-k-m} \binom{s}{k-n} (-1)^{l-k-m}$$

$$= (-1)^{l+m} \sum_{k \le l} \binom{l-k-m-(-m-1)-1}{l-k-m} \binom{s}{k-n} (-1)^{l-k-m}$$

Proving Identity 5.25 (cont.)

Recall that $\binom{r}{k} = (-1)^k \binom{k-r-1}{k}$ was the form for upper negation when k is an integer. Since all our terms in our sum are integers we can let r= -m-1 and k =l-m-k in our sum. Thus allowing us to use upper negation. So we have:

$$(-1)^{l+m} \sum_{k \le l} \binom{l-k-m-(-m-1)-1}{l-k-m} \binom{s}{k-n} (-1)^{l-k-m}$$

$$= (-1)^{l+m} \sum_{k \le l} \binom{-m-1}{l-k-m} \binom{s}{k-n} = (-1)^{l+m} \sum_{k} \binom{-m-1}{l-k-m} \binom{s}{k-n}$$

Since k is in the bottom part of the terms the only restriction on k is that the lower part be integer. Thus we can sum loosen our restriction on k and sum over all k. Now applying Vandermonde's Convolution to the sum we obtain:

$$= (-1)^{l+m} \binom{s-m-1}{l-m-n}$$

Which is the right hand side of we needed to show identity 5.25. ■

Proving Identity 5.26

In order to prove Identity 5.26 we will take the same approach as we did in solving 5.25 but we will use identity 5.25 in the proof in stead of 5.22. Recall the left hand side of identity 5.26

$$\sum_{0\le k\le l}\binom{l-k}{m}\binom{q+k}{n}$$

Since $n \ge q \ge 0$ we can apply symmetry to the second binomial coefficient to obtain:

$$\sum_{0\le k\le l}\binom{l-k}{m}\binom{q+k}{q+k-n}$$

Proving Identity 5.26 (cont.)

Making the substitution k-q for k we can modify the sum to get:

$$\sum_{0\le k-q\le l}\binom{l-(k-q)}{m}\binom{q+(k-q)}{q+(k-q)-n}=\sum_{q\le k\le l+q}\binom{l+q-k}{m}\binom{k}{k-n}$$

Now applying upper negation once again to the left term this time we obtain:

$$\sum_{q\le k\le l+q}\binom{l+q-k}{m}\binom{k-n-k-1}{k-n}(-1)^{k-n}\quad=(-1)^{-n}\sum_{q\le k\le l+q}\binom{l+q-k}{m}\binom{-n-1}{k-n}(-1)^{k}$$

Letting l+q= l' we have:

$$(-1)^{-n}\sum_{k\le l'}\binom{l'-k}{m}\binom{-n-1}{k-n}(-1)^{k}$$

Proving Identity 5.26 (cont.)

Now we have a form similar to 5.25. Letting s=-n-1 and l be l ' we can use our previous identity to get:

$$(-1)^{-n}\sum_{k\le l'}\binom{l'-k}{m}\binom{-n-1}{k-n}(-1)^{k}=(-1)^{-n}(-1)^{l+q+m}\binom{-n-1-m-1}{l+q-m-n}$$

$$=(-1)^{l+q-m-n}\binom{-n-1-m-1}{l+q-m-n}=(-1)^{l+q-m-n}\binom{l+q-m-n-(l+q+1)-1}{l+q-m-n}$$

Using upper negation and symmetry one more time we obtain the left hand side of identity 5.26:

$$=\binom{l+q+1}{l+q-m-n}=\binom{l+q+1}{l+q+1-(m+n+1)}=\binom{l+q+1}{m+n+1}$$