Chapter 5: Problem 15

cse547

Question

- What is $\sum_k \binom{n}{k}^3 (-1)^k$? *Hint: See (5.29).*
- 5.29:

$$\sum_k \binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k}(-1)^k = \frac{(a+b+c)!}{a!\,b!\,c!}$$

Let's consider a few examples

$$\sum_k \binom{n}{k}^3 (-1)^k$$

- **n = 0:** $\binom{0}{0}^3 (-1)^0 = 1 \cdot 1 = 1$
- **n = 1: 1 + (-1) = 0**
  - k = 0: $1^3(-1)^0 = 1$
  - k = 1: $1^3(-1)^1 = -1$
- **n = 2: 1 – 8 + 1 = -6**
  - k = 0: $1^3(-1)^0 = 1$
  - k = 1: $2^3(-1)^1 = -8$
  - k = 2: $1^3(-1)^2 = 1$
- **n = 3: 1 – 27 + 27 – 1 = 0**
  - k = 0: $1^3(-1)^0 = 1$
  - k = 1: $3^3(-1)^1 = -27$
  - k = 2: $3^3(-1)^2 = 27$
  - k = 3: $1^3(-1)^3 = -1$
- **n = 4: 1–64+216–64+1 = 90**
  - k = 0: $1^3(-1)^0 = 1$
  - k = 1: $4^3(-1)^1 = -4^3$
  - k = 2: $6^3(-1)^2 = 6^3$
  - k = 3: $4^3(-1)^3 = -4^3$
  - k = 4: $1^3(-1)^4 = 1$
- **n = 5: = 0**
  - k = 0: $1^3(-1)^0 = 1$
  - k = 1: $5^3(-1)^1 = -5^3$
  - k = 2: $10^3(-1)^2 = 10^3$
  - k = 3: $10^3(-1)^3 = -10^3$
  - k = 4: $5^3(-1)^4 = 5^3$
  - k = 5: $1^3(-1)^5 = -1$

Note: If k > n, then the value will always be zero, since one of the multiplicands is zero, so we don't consider them.

Example Explanation

- It is clear from the previous slide that for all odd n, the resulting summation will be equal to zero.
  - n = 1: 1 + (-1) = 0
  - n = 3: 1 + (-27) + 27 + (-1) = 0
  - n = 5: 1 + (-125) + 1000 + (-1000) + 125 + (-1) = 0
- This is due to the property (5.4): $\binom{n}{k} = \binom{n}{n-k}$
- Since we are multiplying each summand by $(-1)^k$, the values will alternate between positive and negative. They will, therefore, cancel each other out.
- This is only true for odd n's (as we saw on the previous page, because the for the odd n's we end up with an even number of summands.

Let's look at the formulas again:

Want:

$$\sum_k \binom{n}{k}^3 (-1)^k$$

(5.29): with m instead of k

$$\sum_k \binom{a+b}{a+m}\binom{b+c}{b+m}\binom{c+a}{c+m}(-1)^m = \frac{(a+b+c)!}{a!b!c!}$$

- It is obvious from just looking at them, that they look very similar.
- We realize immediately that we want something of the form:
  - n := a + b = b + c = c + a
  - k := a + m = b + m = c + m

Consider our two equations:

- Equations
  - $n := a + b = b + c = c + a$ (1)
  - $k := a + m = b + m = c + m$ (2)
- We have 4 unknowns! But it is easy to see from (2) that we can subtract out m from 3 sides:

$$
\begin{array}{ccccc}
a + m & = & b + m & = & c + m \\
\underline{\;\; - m} & & \underline{\;\; - m} & & \underline{\;\; - m} \\
a & = & b & = & c
\end{array}
$$

Let a := a = b = c

- From this, equation (5.29) would turn into:

$$\sum_k \binom{a+a}{a+m}\binom{a+a}{a+m}\binom{a+a}{a+m}(-1)^m = \frac{(a+a+a)!}{a!a!a!}$$

- By simplifying it further, we get:

$$\sum_m \binom{2a}{a+m}^3 (-1)^m = \frac{(3a)!}{(a!)^3}$$

So can we get to the original formula?

$$\sum_k \binom{n}{k}^3 (-1)^k$$

$$\sum_m \binom{2a}{a+m}^3 (-1)^m = \frac{(3a)!}{(a!)^3}$$

- Hmm… The two are certainly very similar!
- We do have an extra unknown m, however, which also happens to be the base of the summation

So what are our n and k?

$$\sum_{k} \binom{n}{k}^{3} (-1)^{k} \qquad\qquad \sum_{m} \binom{2a}{a+m}^{3} (-1)^{m} = \frac{(3a)!}{(a!)^{3}}$$

- Well… Let’s let k be a+m, and n be 2a

$$\sum_{a+m} \binom{2a}{a+m}^{3} (-1)^{a+m} = \sum_{a+m} \binom{2a}{a+m}^{3} (-1)^{a} (-1)^{m}$$

Let's talk about limits for a second…

$$\sum_{a+m} \binom{2a}{a+m}^3 (-1)^a (-1)^m \qquad \sum_m \binom{2a}{a+m}^3 (-1)^m = \frac{(3a)!}{(a!)^3}$$

- The limit of the summation is a+m.
- So here we have: $a+m \leq 2a$
  - subtract a from both sides
- This is equivalent to: $m \leq a$
- So we have:

This part is exactly like our formula above!

constant

$$\sum_{m \leq a}^{a} \binom{2a}{a+m}^3 (-1)^a (-1)^m = \boxed{\sum_m \binom{2a}{a+m}^3 (-1)^m} \boxed{(-1)^a}$$

The result

- Finally we get:
  - For any n = 2a (n is even)

$$\sum_k \binom{n}{k}^3 (-1)^k = \sum_m \binom{2a}{a+m} (-1)^m (-1)^a = (-1)^a \frac{(3a)!}{(a!)^3}$$

  - For any n, s.t. n is odd:

$$\sum_k \binom{n}{k}^3 (-1)^k = 0$$

- This certainly does seem reasonable!

Check the result:

$$\sum_k \binom{n}{k}^3 (-1)^k = (-1)^a \frac{(3a)!}{(a!)^3}$$

- n = 0 → m = 0
  - $(-1)^0(3*0)! / (0!)^3 = 1$
- n = 2 → m = 1
  - $(-1)^1(3*1)! / (1!)^3 = (-1)(6) / 1 = -6$
- n = 4 → m = 2
  - $(-1)^2(3*2)! / (2!)^3 = (1)(6*5*4*3*2) / (2)^3 = 90$
- etc…
- These are the same as the results we got previously!