Chapter 5 , Problem 17

Find the simple relation between

$\binom{2n-1/2}{n}$ and $\binom{2n-1/2}{2n}$

Using $\binom{n}{k} = \dfrac{n(n-1)(n-2)\ldots\ldots(n-k+1)}{k!}$

$\binom{2n-\frac{1}{2}}{n} = \dfrac{\left(2n-\frac{1}{2}\right)\left(2n-\frac{1}{2}-1\right)\left(2n-\frac{1}{2}-2\right)\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2n-\frac{1}{2}-n+1)}{n(n-1)\ldots.1}$

Both numerator and denominator has n terms

$$= \frac{\left(2n-\frac{1}{2}\right)\left(2-\frac{1}{2}-1\right)\left(2n-\frac{1}{2}-2\right)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2n-\frac{1}{2}-n+1)}{n!}$$

$$= \frac{\left[\frac{(4n-1)}{2}\right]\left[\frac{(4n-3)}{2}\right]\left[\frac{(4n-5)}{2}\right] \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . (n+1/2)}{n!}$$

$$= \frac{\left[\frac{(4n-1)}{2}\right]\left[\frac{(4n-3)}{2}\right]\left[\frac{(4n-5)}{2}\right] \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . \left[\frac{(2n+1)}{2}\right]}{n!}$$

$$= \frac{(4n-1)(4n-3)(4n-5) \ldots \ldots \ldots . (2n+1)}{2^n \, n!}$$

$$= \frac{(4n-1)(4n-3)(4n-5) \ldots \ldots . . (4n-(2n-1))}{2^n \, n!}$$

Multiplying the numerator and denominator by

4n(4n-2)(4n-4).....(4n-(2n-2)) and distributing them in between

$$\frac{4n(4n-1)(4n-2)(4n-3)(4n-4) \ldots . . \big(4n-(2n-2)\big)\big(4n-(2n-1)\big).}{4n(4n-2)(4n-4) \ldots . . (4n-(2n-2)) \;\; 2^n \, n!}$$

--------- (Eq 1)

Now denominator $= 4n(4n-2)(4n-4) \ldots . . (4n-(2n-2)) \;\; 2^n \, n!$

Taking 2 out of each of n terms in RED.

Denominator =

$$2^n \, 2n(2n-1)(2n-2) \ldots . . (2n-(n-1))) \;\; 2^n \, n!$$

$$2^{2n} \, 2n(2n-1)(2n-2) \ldots . . (2n-(n-1))) \;\; n!$$

$2^{2n}\ 2n(2n-1)(2n-2)\ \ldots..(n+1)\ \ n!$

$2^{2n}\ (2n)!$

Now Substituting the value of denominator in Eq(1)

$$=\frac{4n(4n-1)(4n-2)(4n-3)(4n-4)\ \ldots..\big(4n-(2n-2)\big)(4n-2n+1).}{2^{2n}\ (2n)!}$$

$$=\frac{\binom{4n}{2n}.}{2^{2n}}$$

Hence $\binom{2n-\frac{1}{2}}{n}$ = $\frac{\binom{4n}{2n}.}{2^{2n}}$

Similarly we need to find $\binom{2n-\frac{1}{2}}{2n}$ =

$$=\frac{\left(2n-\frac{1}{2}\right)\left(2-\frac{1}{2}-1\right)\left(2n-\frac{1}{2}-2\right)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ 2n-\frac{1}{2}-2n+1)}{(2n)!}$$

$$=\frac{\left[\frac{(4n-1)}{2}\right]\left[\frac{(4n-3)}{2}\right]\left[\frac{(4n-5)}{2}\right]\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(1/2)}{2n)!}$$

$$= \frac{\left[\frac{(4n-1)}{2}\right]\left[\frac{(4n-3)}{2}\right]\left[\frac{(4n-5)}{2}\right]\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\left[\frac{1}{2}\right]}{(2n)!}$$

Since we have 2n terms in numerator

$$= \frac{(4n-1)(4n-3)(4n-5)\ldots\ldots\ldots.(1)}{2^{2n}\,(2n)!}$$

$$\frac{(4n-1)(4n-3)(4n-5)\ldots\ldots..(1))}{2^{2n}\,(2n)!}$$

Multiplying the numerator and denominator by

4n(4n-2)(4n-4).....(2) and distributing them in between the numerator in above equation

$$\frac{4n(4n-1)(4n-2)(4n-3)(4n-4)\ldots..(2)(1).}{4n(4n-2)(4n-4)\ldots..(2)\,2^{2n}\,(2n)!}$$

--(Eq 2)

Now denominator =$[4n(4n-2)(4n-4)\ldots..\left(4n-(4n-2)\right)]\,2^{2n}\,(2n)!$

Taking 2 out of each of (2n) terms in RED.

Denominator =

$2^{2n}\,2n(2n-1)(2n-2)\ldots..(2n-(2n-1)))\;\;2^{2n}\,(2n)!$

$2^{2n}\,2n(2n-1)(2n-2)\ldots..(1)\;\;2^{2n}\,(2n)!$

$2^{2n}(\,2n)!\;\;2^{2n}\,(2n)!$

$2^{4n}\,(2n)!\;\;(2n)!$

Substituting the value of the above denominator in Eq (2)

$$\frac{4n(4n-1)(4n-2)(4n-3)(4n-4)\,\ldots..(2)(1).}{2^{4n}\,(2n)!\;\;(2n)!}$$

$$\frac{(4n)!}{2^{4n}\,(2n)!\;\;(2n)!}$$

Hence $\binom{2n-\frac{1}{2}}{2n} = \dfrac{(4n)!}{2^{4n}\,(2n)!\;\;(2n)!}$

Combining the solutions we get

$$\frac{\binom{2n-\frac{1}{2}}{n}}{\binom{2n-\frac{1}{2}}{2n}} = \frac{\frac{\binom{4n}{2n}.}{2^{2n}}}{\frac{(4n)!}{2^{4n}\,(2n)!\;(2n)!}} = \frac{\frac{(4n)!}{2^{2n}\,(2n)!\;(2n)!}}{\frac{(4n)!}{2^{4n}\,(2n)!\;(2n)!}} = 2^{2n}$$

Chapter 5 Problem 16

Evaluate the sum

$$\sum_k \binom{2a}{a+k}\binom{2b}{b+k}\binom{2c}{c+k}(-1)^k$$

Using formula

$$\binom{n}{k} = \frac{n!\,(n-k)!}{k!}$$

For $\binom{2a}{a+k}$,$\binom{2b}{b+k}$, $\binom{2c}{c+k}$ we have,

$$\sum_k \binom{2a}{a+k}\binom{2b}{b+k}\binom{2c}{c+k}(-1)^k$$

$$= \sum_k \frac{2a!}{(a+k)!\,(a-k)!}\frac{2b!}{(b+k)!\,(b-k)!}\frac{2c!}{(c+k)!\,(c-k)!}(-1)^k$$

$$= \sum_k \frac{2a!}{(a+k)!\,(a-k)!} \frac{2b!}{(b+k)!\,(b-k)!} \frac{2c!}{(c+k)!\,(c-k)!} (-1)^k$$

We notice that

In order to solve the summation using equation 5.29, we must convert the summand terms to the product

$\binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k}$

Thus consider

$$\binom{a+b}{a+k} = \frac{(a+b)!}{(a+k)!\,(b-k)!}$$

Similarly

$$\binom{b+c}{b+k} = \frac{(b+c)!}{(b+k)!\,(c-k)!}$$

$$\binom{a+c}{c+k} = \frac{(a+c)!}{(c+k)!\,(a-k)!}$$

Multiplying these 3 terms we get

$$\binom{a+b}{a+k}\binom{b+c}{b+k}\binom{a+c}{c+k}$$
$$= \frac{(a+b)!\ (b+c)!\ (a+c)!}{(a+k)!\,(b-k)!\,(b+k)!\,(c-k)!\,(c+k)!\,(a-k)!}$$

We observe that denominator of our equation exactly matches denominator of our required equation. To match the numerator we multiply the equation by

$$\frac{(a+b)!\,(b+c)!\,(a+c)!}{(a+b)!\,(b+c)!\,(a+c)}$$

So we have

$$\sum_k \binom{2a}{a+k}\binom{2b}{b+k}\binom{2c}{c+k}(-1)^k$$

$$= \sum_k \frac{(2a)!\,(a+b)!}{(a+k)!\,(a-k)!\,(a+b)!}\frac{(2b)!\,(b+c)!}{(b+k)!\,(b-k)!\,(b+c)!}\frac{(2c)!\,(a+c)!}{(c+k)!\,(c-k)!\,(a+c)!}(-1)^k$$

As the terms are independent of k we can take them out of the summation.

$$\frac{(2a)!\,(2b)!\,(2c)!}{(a+b)!\,(b+c)!\,(a+c)}$$

$$= \frac{(2a)!\,(2b)!\,(2c)!}{(a+b)!\,(b+c)!\,(a+c)!}\sum_k \frac{(a+b)!}{(a+k)!\,(a-k)!}\frac{(b+c)!}{(b+k)!\,(b-k)!}\frac{(c+k)!}{(c+k)!\,(c-k)!}(-1)^k$$

Note that

$$\frac{(a+b)!}{(a+k)!\,(b-k)!} = \binom{a+b}{a+k}$$

$$\frac{(b+c)!}{(b+k)!(c-k)!} = \binom{b+c}{b+k}$$

$$\frac{(a+c)!}{(c+k)!(a-k)!} = \binom{a+c}{c+k}$$

$$= \frac{(2a)!(2b)!(2c)!}{(a+b)!(b+c)!(a+c)!} \sum_k \binom{a+b}{a+k}\binom{b+c}{b+k}\binom{a+c}{a+k}(-1)^k$$

From book we have equation 5.29

$$\sum \binom{a+b}{a+k}\binom{b+c}{b+k}\binom{a+c}{a+k}(-1)^k = \frac{(a+b+c)!}{a!\,b!\,c!}$$

Thus by substituting the formula,

$$= \frac{(2a)!(2b)!(2c)!}{(a+b)!(b+c)!(a+c)!} \frac{(a+b+c)!}{a!\,b!\,c!}$$

So finally we have

$$\sum_k \binom{2a}{a+k}\binom{2b}{b+k}\binom{2c}{c+k}(-1)^k$$
$$= \frac{(2a)!(2b)!(2c)!}{(a+b)!(b+c)!(a+c)!} \frac{(a+b+c)!}{a!\,b!\,c!}$$