Cse547
Chapter 5 problem 18

Problem 18

- Find an alternative form analogous to (5.35)

  for the product $\binom{r}{k}\binom{r-1/3}{k}\binom{r-2/3}{k}$

- From the textbook, we can see

$$\binom{r}{k}\binom{r-1/2}{k} = \binom{2r}{2k}\binom{2k}{k}\Big/ 2^{2k} \quad (5.35)$$

Let's review this

$$\binom{r}{k} = \frac{r(r-1)(r-2)\cdots(r-k+1)}{k!} = \frac{r!}{(r-k)!k!}$$

So, now let's go to the question

$$\binom{r}{k}\binom{r-1/3}{k}\binom{r-2/3}{k}$$

$$= \frac{r(r-1)\cdots(r-k+1)}{k!}\frac{(r-1/3)\cdots(r-1/3-k+1)}{k!}\frac{(r-2/3)\cdots(r-2/3-k+1)}{k!}$$

$$= \frac{r(r-1)\cdots(r-k+1)}{k!}\frac{(r-1/3)\cdots(r-k+2/3)}{k!}\frac{(r-2/3)\cdots(r-k+1/3)}{k!}$$

$$= \frac{r\left(r-\frac{1}{3}\right)\left(r-\frac{2}{3}\right)\left(r-\frac{3}{3}\right)\cdots(r-k+1)\left(r-k+\frac{2}{3}\right)\left(r-k+\frac{1}{3}\right)}{k!k!k!}$$

$$\frac{r\left(r-\frac{1}{3}\right)\left(r-\frac{2}{3}\right)\left(r-\frac{3}{3}\right)\cdots(r-k+1)\left(r-k+\frac{2}{3}\right)\left(r-k+\frac{1}{3}\right)}{k!k!k!}$$

Multiply by 3^{3k}

$$=\frac{3r(3r-1)(3r-2)(3r-3)\cdots(3r-3k+3)(3r-3k+2)(3r-3k+1)}{k!k!k!}\cdot\frac{1}{3^{3k}}$$

$$=3r(3r-1)(3r-2)(3r-3)\cdots(3r-3k+1)\cdot\frac{1}{k!k!k!\cdot 3^{3k}}$$

$$=\frac{(3r)!}{(3r-3k)!}\cdot\frac{1}{k!k!k!\cdot 3^{3k}}$$

Now we need some thing like $\binom{3r}{3k}\binom{3k}{??}\cdots \Big/ 3^{3k}$

$$= \frac{(3r)!}{(3r-3k)!} \cdot \frac{1}{k!k!k! \cdot 3^{3k}}$$

$$= \frac{(3r)!}{(3r-3k)!(3k)!} \cdot \frac{(3k)!}{(2k)!} \cdot \frac{(2k)!}{1} \cdot \frac{1}{k!k!k! \cdot 3^{3k}}$$

$$= \frac{(3r)!}{(3r-3k)!(3k)!} \cdot \frac{(3k)!}{(2k)!k!} \cdot \frac{(2k)!}{k!k!} \cdot \frac{1}{3^{3k}}$$

$$= \binom{3r}{3k}\binom{3k}{2k}\binom{2k}{k} \Big/ 3^{3k}$$

Finally, we have the solution

$$\binom{r}{k}\binom{r-1/3}{k}\binom{r-2/3}{k} = \binom{3r}{3k}\binom{3k}{2k}\binom{2k}{k} \Big/ 3^{3k}$$