BINOMIAL THEOREM
BINOMIAL COEFFICIENTS

$$(x+y)^0 = 1 \cdot x^0 \cdot y^0$$

$$(x+y)^1 = 1 \cdot x^1 \cdot y^0 + 1 \cdot x^0 y^1$$

$$(x+y)^2 = 1 \cdot x^2 y^0 + 2 x^1 y^1 + 1 \cdot x^0 y^2$$

$$(x+y)^3 = 1 \cdot x^3 \cdot y^0 + 3 x^2 y^1 + 3 x^1 y^2 + 1 x^0 y^3$$

$$(x+y)^n = \underbrace{(x+y)(x+y) \cdots (x+y)}_{n\text{-times}}$$

How many are there terms $x^k y^{n-k}$?

As many as number of ways to choose k of the n-binomials from which an x will be contributed

i.e $\binom{n}{k}$

We define $\boxed{x, y \in \mathbb{R}.}$

$x^0 = 1$ for all x

in particular the case $x=0, y=0$ or $y=-x$, we get case $0^0=1$, $n=0$

$n \in \mathbb{R}.$ $|x/y|<1$

BINOMIAL THEOREM

$$(x+y)^n = \sum_k \binom{n}{k} x^k y^{n-k}$$

or

$n \in \mathbb{N}$
$k \in \mathbb{Z}$

but terms $=0$ except for $0 \le k \le n$

$$(x+y)^{\boxed{n}} = \sum_{k=0}^{\boxed{n}} \binom{n}{k} x^k y^{n-k}$$

Theorem is valid also when when $n \in \mathbb{R}$. In this case $\sum_k$ is infinite and we must have $\boxed{|x/y|<1}$ to guarantee absolute convergence.

TWO SPECIAL CASES

OF

$$\boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}}$$

CASE 1

$x = y = 1$, $\boxed{n \in N}$

$$\binom{0}{0} = 1$$

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k \cdot 1^{n-k}$$

Extra

$$\boxed{\binom{n+1}{k+1} = 2^k}$$

$\neq$ $^{k+1}$

$$\boxed{2^n = \sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \cdots \binom{n}{n}}$$

$$\sum_{k=0}^{n} \binom{m}{k} \quad m \neq n$$

no formula

CASE 2

$x = -1$, $y = 1$

$$0^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$$

$$\boxed{0^n = \binom{n}{0} - \binom{n}{1} + \cdots (-1)^n \binom{n}{n}}$$

$n \in N$

$$\boxed{\begin{array}{c} n = 0 \\ 0^0 = 1 \end{array}}$$

Ex. $1 - 4 + 6 - 4 + 1 = 0^4 = 0$

$0^n = 0 \quad n > 0$

CASE 3 $\quad x \in R,\ y = 1,\ n \in N$ or $x \in$ Comp

$$\boxed{(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k}$$

IN GENERAL $\quad z \in R,\ k \in Z,\ r \in Z$

or $z \in$ Complex

$$\boxed{(1+z)^r = \sum_{k} \binom{r}{k} z^k}$$

$\boxed{|z| < 1}$

for convergence
we do for
$z \in R$

CONSIDER

$$\boxed{\begin{aligned} & f : R \to R \\ & f(z) = (1+z)^r \end{aligned}}$$

$r \in N$

$r \in R$

$$f'(z) = r(1+z)^{r-1}, \quad f''(z) = r(r-1)(1+z)^{r-2}$$

$$f^k(z) = r(r-1)\ldots(r-k+1)(1+z)^{r-k}$$

$$\boxed{f^k(z) = r^{\underline{k}}(1+z)^{r-k}} \qquad \boxed{f^k(0) = r^{\underline{k}}}$$

TO PROVE USE

Assume $f \in \mathrm{Diff}^\infty$

TAYLOR CALCULUS THM

$$\boxed{f(z) = \sum_{k \geq 0} \frac{f^k(0)}{k!} z^k}$$

TYLOR

$$\boxed{f(a+x) = \sum_{k \geq 0} \frac{f^k(a)}{k!} x^k}$$

Take $f(z) = (1+z)^r$, ~~z=0~~ $f^k(0) = r^{\underline{k}}$ $r \in \mathbb{R}$

WE GET:

$$\boxed{(1+z)^r = \sum_{k \geq 0} \frac{r^{\underline{k}}}{k!} z^k = \sum_{k \geq 0} \binom{r}{k} z^k}$$

and it converges when $|z| < 1$

~~Assume~~

SOME CASES OF $\binom{n}{k}$ for $n < 0$

Evaluate $\binom{-1}{0}$

use $\boxed{\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}}$

$$\binom{0}{0} = \binom{-1}{0} + \binom{-1}{-1}$$

$$1 = \binom{-1}{0} + 0$$

$$\boxed{\binom{-1}{0} = 1}$$

Evaluate

$\binom{-1}{1}$ using $\boxed{\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}}$

$$\binom{0}{1} = \binom{-1}{1} + \binom{-1}{0}$$

$$0 = \binom{-1}{1} + 1$$

$$\boxed{\binom{-1}{1} = -1}$$

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Pascal Triangle, extended upward

n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$
-4	1	-4	10	-20	35	-56
-3	1	-3	6	-10	15	-21
-2	1	-2	3	-4	5	-6
-1	1	-1	1	-1	1	-1
0	1	0	0	0	0	0

NEGATING UPPER LIMIT

General Rule

$$\boxed{\binom{r}{k} = (-1)^k \binom{k-r-1}{k}} \qquad k \in \mathbb{Z}$$

Formula is true even when $r \in \mathbb{R}$

$$\boxed{\binom{x}{k} = (-1)^k \binom{k-x-1}{k}} \; ① \qquad \begin{array}{l} x \in \mathbb{R} \\ k \in \mathbb{Z} \end{array}$$

Proof

$$x^{\underline{k}} = x(x-1)\ldots(x-k+1)$$

$$= (-1)^k (-x)(1-x)(2-x)\ldots(k-1-x)$$

$$= \boxed{(-1)^k (k-x-1)^{\underline{k}}}$$

Evaluate

$$(k-x-1)^{\underline{k}} = (k-x-1)(k-x-\cancel{1}+\cancel{1})\ldots$$

$$\underbrace{(k-x-1-k+1)}_{(-x)}$$

REMARK (negate twice!)

$$\binom{x}{k} \overset{①}{=} (-1)^k \binom{k-x-1}{k} =$$

$$\overset{①}{=} (-1)^k (-1)^k \binom{k-(k-x-1)-1}{k}$$

$$= (-1)^{2k} \binom{x}{k} = \binom{x}{k}$$

①

$$\boxed{\binom{x}{k} = (-1)^k \binom{k-x-1}{k}}$$

SYMMETRY

②

$$\boxed{(-1)^m \binom{-n-1}{m} = (-1)^n \binom{-m-1}{n}}$$

Proof from ①

$$\text{LEFT} = (-1)^m \binom{\overset{n}{-n-1}}{\underset{k}{m}} = (-1)^m (-1)^m \binom{m-(-n-1)-1}{m}$$

$$= (-1)^{2m} \binom{m+n}{m} = \binom{m+n}{m}$$

$$\boxed{\text{LEFT} = \binom{m+n}{m}}$$

$$\text{RIGHT} = (-1)^n \binom{\overbrace{-m}^{x}-1}{n}$$

$$\boxed{\binom{x}{k} = (-1)^k \binom{k-x-1}{k}} \quad (1)$$

$$\overset{(1)}{=} (-1)^n(-1)^n \binom{n-(-m-1)-1}{n}$$

$$= (-1)^{2n} \binom{n+m}{n}$$

$$\boxed{\text{RIGHT} = \binom{n+m}{m}}$$

LEFT = RIGHT means that

$$\boxed{\binom{n+m}{m} = \binom{n+m}{n}}$$

for $n, m \in \mathbb{Z}$

Proof: use $\boxed{\binom{n}{k} = \binom{n}{n-k}}$

$$\binom{n+m}{m} = \binom{n+m}{n+m-m} = \binom{n+m}{n}$$

yes!

EVALUATE:

$$\sum_{k \le m} \binom{x}{k}(-1)^k = \binom{x}{0} - \binom{x}{1} + \dots + (-1)^m \binom{x}{m}$$

$k<0$ terms $=0$

SHOW

$$\boxed{\sum_{k \le m} \binom{x}{k}(-1)^k = (-1)^m \binom{x-1}{m}}$$

$m \in \mathbb{Z}$

$x \in \mathbb{R}$

① $\boxed{\binom{x}{k} = (-1)^k \binom{k-x-1}{k}}$

LEFT:

$$\sum_{k \le m} \binom{x}{k}(-1)^k \overset{①}{=} \sum_{k \le m} (-1)^k (-1)^k \binom{k-x-1}{k}$$

$$= \sum_{k \le m} \binom{k-x-1}{k} \rightarrow$$

use

$$\boxed{\sum_{k \le m} \binom{x+k}{k} = \binom{x+m+1}{m}}$$

$$= \sum_{k \le m} \binom{\overbrace{-x-1}^{x} + k}{k} = \binom{-x-1+m+1}{m}$$

$$= \binom{\overbrace{-x+m}^{x}}{m} \overset{①}{=} (-1)^m \binom{m-(-x+m)-1}{m} = (-1)^m \binom{x-1}{m}$$

$=$ RIGT