REMARK:

we prove

$$\boxed{\forall m \quad J(2^m) = 1}$$

Induction

Use: $J(2n) = 2J(n) - 1$

$$\boxed{J(2^m)} = J(\underbrace{2 \cdot 2^{m-1}}_{2n}) = 2 \cdot J(\overbrace{2^{m-1}}^{n}) - 1$$

by ind ass. $= 1$

IND ASSUMP $= 2 \cdot 1 - 1 = 1$

HENCE:

THEOREM

FIRST PERSON will always survive whenever n is a power of 2.

GENERAL CASE: $n = 2^m + l$

the number of people is reduced to power of 2 AFTER there have been l executions.

The first remaining person, the SURVIVOR is number $2l+1$

SOLUTION : MORE ABOUT IT !

$$\boxed{J(2^m+l)=2l+1} \qquad n=2^m+l \quad 0\le l<2^m$$

depends heavily on POWERS of 2.

Let's look at

BINARY EXPENTION of n

$$n=(b_m b_{m-1} \dots b_1 b_0)_2$$

stands for

$$n=b_m 2^m + b_{m-1} 2^{m-1} + \dots b_1 2 + b_0$$

for $b_i \in \{0,1\}$, $b_m=1$

For example

$$n=100$$

$$\boxed{n=(1100100)_2}$$

$2^6\ 2^5 \dots 2^2\ 2^1\ 2^0$

$$n=2^6+2^5+2^2=64+32+4=100$$

Observe :

$$n = 2^m + l \quad , \quad 0 \le l < 2^m$$

and we have the following binary expentions :

① $\boxed{l = (0, b_{m-1}, \dots b_1, b_0)_2}$ as $l < 2^m$

② $\boxed{2l = (b_{m-1}, \quad b_1, b_0, 0)_2}$ as

$$l = b_{m-1} 2^{m-1} + \dots + b_1 2 + b_0$$

$$2l = b_{m-1} 2^m + \qquad b_1 2^2 + b_0 2 + 0$$

③ $2^m = (1, 0, \dots \dots 0)_2 \ , \quad 1 = (0 \dots \dots 1)_2$

④ $n = 2^m + l$

$\boxed{n = (1, b_{m-1} \dots b_1 b_0)_2}$ ① + ③

⑤ $\boxed{2l + 1 = (b_{m-1} \, b_{m-2} \dots b_0 \, 1)_2}$ ② + ③

CLOSED FORMULA

$$\boxed{J(n) = 2\ell + 1} \quad \text{for } n = 2^m + \ell$$

We re-write it in binary expansion as follows

$$\boxed{J((b_m, b_{m-1} \ldots b_1 b_0)_2) = (b_{m-1} \ldots b_1 b_0 b_m)_2}$$

because $b_m = 1$ in binary exp. of n

Example: FIND $J(100)$

$n = 100 = (1100100)_2$

$J(100) = (1001001)_2$

$J(100) = 64 + 8 + 1 = 73$

$$J((1, b_{m-1}, \ldots b_1, b_0)_2) = (b_{m-1}, \ldots b_1, b_0, 1)_2$$

GENERALIZATION

Our function J was defined (R)

$J: N \to N$

$$J(1) = 1$$

$$J(2n) = 2J(n) - 1 \qquad n \geq 1$$

$$J(2n+1) = 2J(n) + 1 \qquad n \geq 1$$

We generalize it as follows (R)

$$\boxed{\begin{aligned} f(1) &= \alpha \\ f(2n) &= 2f(n) + \beta \qquad n \geq 1 \\ f(2n+1) &= 2f(n) + \gamma \qquad n \geq 1 \end{aligned}}$$

$\boxed{f: N \to N}$

$\boxed{J = f}$

for $\alpha = 1$, $\beta = -1$, $\gamma = +1$

PROBLEM: $\boxed{\text{Find a CLOSED form of } f.}$

PROBLEM

Given R → find CF

for

$$\boxed{f(1)=\alpha,\quad f(2n)=2f(n)+\beta,\quad f(2n+1)=2f(n)+\gamma}$$

STEP 1 Find few initial values for f

STEP 2 Find (guess) a CF formula from STEP 1

GENERAL FORM for CF

STEP 3 Prove CF formula

CF = R

STEP 3 is usually done by math induction (domain = N !)

HERE we present a

REPERTOIRE METHOD

→ OUT exact form from ②

we examine a repertoire of cases and use it to find general CF formula

$\boxed{\text{STEP 1}}$: evaluate few values for

$$\boxed{f(1) = \alpha\,, \quad \begin{aligned} f(2n) &= 2f(n) + \beta \\ f(2n+1) &= 2f(n) + \gamma \end{aligned}}$$

n	$f(n)$	Calculation
1	α	$f(1) = \alpha$
2	$2\alpha + \beta$	$f(2) = 2f(1) + \beta$
3	$2\alpha + \gamma$	$f(3) = 2f(1) + \gamma$
4	$4\alpha + 3\beta$	$f(4) = 2f(2) + \beta$
5	$4\alpha + 2\beta + \gamma$	$f(5) = 2f(2) + \gamma$
6	$4\alpha + \beta + 2\gamma$	$f(6) = 2f(3) + \beta$
7	$4\alpha + 3\gamma$	$f(7) = 2f(3) + \gamma$
8	$8\alpha + 7\beta$	$f(8) = 2f(4) + \beta$
9	$8\alpha + 6\beta + \gamma$	$f(9) = 2f(4) + \gamma$

OBSERVATIONS : given $n = 2^k + l$ $0 \le l < 2^k$

α coefficient is 2^k

β c. for the groups decreases by 1 down to 0

β coeficient $2^k - 1 - l$

γ coeficient increases by 1 up from 0

γ coeficient is l

General PROPOSITION :

$$\boxed{f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma}$$

for

$$\boxed{\begin{array}{ll} A(n) = 2^k & \\ B(n) = 2^k - 1 - l & C(n) = l \end{array}}$$

Exersice: Prove by induction over k

We know $f: N \to N$

R $\quad \boxed{f(1)=\alpha, \quad f(2n)=2f(n)+\beta, \quad f(2n+1)=2f(n)+\gamma}$

and

GENERAL

CF $\quad \boxed{f(n)=A(n)\alpha+B(n)\beta+C(n)\gamma}$

EXACT

SHOW $\quad \boxed{A(n)=2^k, \quad B(n)=2^k-1-l, \quad C(n)=l}$

for any $n=2^k+l, \quad 0 \le l < 2^k$

STEP 1 Consider case $\boxed{\alpha=1, \; \beta=\gamma=0}$

R $\quad f(1)=1, \quad f(2n)=2f(n), \quad f(2n+1)=2f(n)$

CF put $\boxed{f(n)=A(n)}$

We re-write R in terms of $A(n)$:

AR $\quad \boxed{A(1)=1, \quad A(2n)=2A(n), \quad A(2n+1)=2A(n)}$

$\boxed{\text{FACT 1}}$ Closed formula CAR for AR is:

CAR $\boxed{A(n) = 2^k}$ for $n = 2^k + l$
$0 \le l < 2^k$

Proof: Use AR and show by induction on k that closed CAR is $2^k = A(n)$

BASE: $\boxed{k=0}$ i.e $n = 2^0 + l$ $0 \le l < 1$, $\to l=0$ $\boxed{n=1}$

AR: $A(1) = 1$ CAR: $A(1) = 2^0 = 1$ yes

$\boxed{\text{ASSUME}}$: $A(2^{k-1} + l) = 2^{k-1}$, $0 \le l < 2^{k-1}$

PROOF: $A(2^k + l) = 2^k$, $0 \le l < 2^k$

$\boxed{\text{TWO CASES}}$: $n \in EVEN$, $n \in ODD$

(1) $n \in EVEN$: $n = 2n$ $\boxed{2^k + l = 2n \text{ iff } l \in EVEN}$ l for $2n$

$n = 2^{k-1} + l/2$

Evaluate l for n k:

$$A(2n) = A(2^k + l) = 2\,A(2^{k-1} + l/2)$$

$$\text{ind assum} = 2 \cdot 2^{k-1} = 2^k \text{ yes}$$

(2) $n = 2n+1$, $2^k + l = 2n+1$, $n = 2^{k-1} + (l-1)/2$

$$A(2n+1) = \boxed{A(2^k + l)} = 2\,A(2^{k-1} + (l-1)/2) = \boxed{2^k} \text{ yes}$$

STEP 2:

Consider a CONSTANT function

(*) $\boxed{f(n)=1}$ all $n\in N$

and evaluate α, β, γ for it (if possible)

(R) $\boxed{f(1)=\alpha,\quad f(2n)=2f(n)+\beta,\quad f(2n+1)=2f(n)+\gamma}$

solution $\quad 1=2+\beta \qquad 1=2+\gamma$

$$\boxed{\alpha=1,\quad \beta=-1,\quad \gamma=-1}$$

(CF) $\boxed{f(n)=A(n)\alpha+B(n)\beta+C(n)\gamma}$

we evaluate (CF) for α, β, γ being solutions for (R) and $\boxed{f(n)=1}$ all n

and get

FACT 2

$$\boxed{A(n)-B(n)-C(n)=1}$$

for all $n\in N$

$\boxed{\text{STEP 3}}$

Consider a "CONSTANT" function

$\boxed{f(n) = n}$ for all $n \in N$

and evaluate α, β, γ for it
(if possible)

R $\boxed{f(1) = \alpha, \quad f(2n) = 2f(n) + \beta, \quad f(2n+1) = 2f(n) + \gamma}$

Solutions: $\quad 2n = 2n + \beta \qquad 2n+1 = 2n + \gamma$

$$\boxed{\alpha = 1 \quad , \quad \beta = 0 \quad , \quad \gamma = 1}$$

CF $\quad f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma$

Evaluate CF for the solutions
$\alpha = 1, \beta = 0, \gamma = 1$ and $f(n) = n$
and we get

$\boxed{\text{FACT 3}}$ $\quad \boxed{A(n) + C(n) = n}$

all $n \in N$

STEP 4

We put together FACT 1, 2 and 3 to evaluate CLOSED formulas for $A(n)$, $B(n)$ and $C(n)$

FACT 1: $A(n) = 2^k$ (CFA)

$\boxed{n = 2^k + l}$, $0 \le l < 2^k$

gives (via FACT 1)

FACT 3: $A(n) + C(n) = n$

$$2^k + C(n) = 2^k + l$$

CFC $\boxed{C(n) = l}$

FACT 2: $A(n) + C(n) - B(n) = 1$

$$2^k + l - B(n) = 1$$

CFB $\boxed{B(n) = 2^k - 1 + l}$

CFA, CFC, CFB are exactly formulas we have guessed!

END

GENERALIZED JOSEPHUS c. d

We proved that the original J-reccurence

$$\boxed{\begin{aligned} &J(1) = 1 \quad \text{and } n \geq 1: \\ &J(2n) = 2J(n) - 1 \\ &J(2n+1) = 2J(n) + 1 \end{aligned}}$$

has a beautiful BINARY SOLUTION

$$\boxed{J(b_m, b_{m-1}, \ldots b_1, b_0)_2 = (b_{m-1}, \ldots b_0, b_m)_2}$$

where $b_m = 1$, as $n = 2^m + l$

QUESTION: DOES the Generalized J admitt a similar solution?

Answer: YES (to follow)

$\boxed{\text{GENERALIZED Josephus}}$ c.d

FIRST : we re-write the G.J. function

$$\boxed{\begin{aligned} &f(1) = \alpha ; \\ &f(2n) = 2f(n) + \beta \quad , n \geq 1 \\ &f(2n+1) = 2f(n) + \gamma \quad , n \geq 1 \end{aligned}}$$

AS FOLLOWS: (more general, yet)

$$\boxed{\begin{aligned} &f(1) = \alpha \\ &f(2n + j) = 2f(n) + \beta_j \\ &\text{for } j = 0, 1 \ , \quad n \geq 1 \end{aligned}}$$

We get GJ function for

$\boxed{\beta_0 = \beta \qquad \beta_1 = \gamma}$ $\qquad \boxed{b_m = 1}$

Assume $\boxed{n = (b_m, b_{m-1}, \ldots b_1, b_0)_2}$

We want to evaluate

$f(n) = \boxed{f(b_m, b_{m-1}, \ldots b_1, b_0)_2}$

G3

Consider case when

$\cancel{k} = 2n + 0$ $\quad \boxed{(j = 0)}$

The binary representation of $2n$ is

$\boxed{2n = (b_m, b_{m-1} \dots b_1, b_0)_2}$ i.e

$$2n = 2^m b_m + \dots + 2b_1 + b_0$$

We get $\boxed{b_m = 1}$ and $\boxed{b_0 = 0}$

Hence

$\boxed{n = 2^{m-1} b_m + \dots + b_1}$ i.e

$\boxed{n = (b_m, b_{m-1}, \dots b_1)}$

What happens when

$\boxed{K = 2n + 1}$ $\quad$ $j = 1$

$$\boxed{2n+1 = (b_m, b_{m-1}, \dots b_1, b_0)_2}$$

$$2n+1 = 2^m b_m + \dots + 2b_1 + b_0$$

and $\boxed{b_0 = 1}$

we get $\boxed{b_m = 1}$

$$2n+1 = 2^m b_m + \dots + 2b_1 + 1$$

$$2n = 2^m b_m + \dots + 2b_1$$

$$n = 2^{m-1} b_m + \dots + b_1$$

$$\boxed{n = (b_m, b_{m-1}, \dots, b_1)}$$

We proved: ~~for any~~
we don't need to consider
cases of n even, or odd. in
$f(2n+j) = 2f(n) + \beta_j$ i.e

G5

Evaluate:

$\boxed{b_m = 1}$

$f((b_m, b_{m-1}, \ldots, b_1, b_0)_2) =$

$= 2f((b_m, \ldots b_1)_2) + \beta_{b_0}$

$= 2(2f((b_m, \cdots b_2)_2) + \beta_{b_1}) + \beta_{b_0}$ where $\boxed{\beta_{b_0} = \begin{cases} \beta_1 & b_0 = 1 \\ \beta_0 & b_0 = 0 \end{cases}}$

$= 4f((b_m, \cdots b_2)_2) + 2\beta_{b_1} + \beta_{b_0}$

$\boxed{\text{where } \beta_{b_1} = \begin{cases} \beta_1 & b_1 = 1 \\ \beta_0 & b_1 = 0 \end{cases}}$

⋮ repeat

$= 2^m f((b_m)_2) + 2^{m-1}\beta_{b_{m-1}} + \ldots + 2\beta_{b_1} + \beta_{b_0}$

$= 2^m f(1) + 2^{m-1}\beta_{m-1} + \cdots + \beta_{b_0}$ where $\boxed{\beta_{b_j} = \begin{cases} \beta_1 & b_j = 1 \\ \beta_0 & b_j = 0 \end{cases}}$

$j = 0, 1, \ldots m-1$

$f(1) = \alpha$ and we got:

$$f((b_m, b_{m-1}, \ldots b_1, b_0)_2) = (\alpha, \beta_{b_m}, \beta_{b_{m-1}}, \ldots \beta_{b_1}, \beta_{b_0})_2$$

More general "binary"

where

$$\beta_{b_j} = \begin{cases} \beta_1 & b_j = 1 \\ \beta_0 & b_j = 0 \end{cases}$$

β_1, β_0 can be any integers

for $j = 0, 1, \ldots m$

General Josephus

$$f(1) = \alpha$$
$$f(2n+j) = 2f(n) + \beta_j$$
$$j = 0, 1, \quad n \geq 1$$

EXAMPLE : Original Josephus

$$f(1) = 1$$

$$f(2n+j) = 2f(n) + \beta_j$$

$j = 0, 1, \quad n \geq 0$

and

$$\boxed{\begin{aligned} \beta_0 &= -1 \\ \beta_1 &= 1 \end{aligned}}$$

$$f(2n) = 2f(n) - 1$$

$$f(2n+1) = 2f(n) + 1$$

$$\boxed{f(100) = 73}$$

As before!

Take $n = 100$

$$\boxed{n = (\overset{b_6}{1}\ \overset{b_5}{1}\ \overset{b_4}{0}\ \overset{b_3}{0}\ \overset{b_2}{1}\ \overset{b_1}{0}\ \overset{b_0}{0})_2}$$

$$\boxed{\beta_{b_j} = \begin{cases} \beta_1 & b_j = 1 \\ \beta_0 & b_j = 0 \end{cases} = \begin{cases} 1 & b_j = 1 \\ -1 & b_j = 0 \end{cases}}$$

$$f(n) = (\alpha, \beta_{b_6}, \beta_{b_5}, \dots \beta_{b_0})$$

$$\boxed{f(n) = (1, 1, -1, -1, 1, -1, -1)_2}$$

$$+64 + 32 - 16 - 8 + 4 - 2 - 1 = 73$$

Prove that the cyclic-shift property holds i.e

$\beta_1 = 1 \quad \beta_0 = -1$

$b_m = 1$

$$\boxed{f((b_m, b_{m-1}, \dots b_1, b_0)_2) = (b_{m-1}, \dots b_0, b_m)}$$

As we knew $f(m) = (1, \beta_{b_m}, \dots \beta_{b_0})$

contains 1, and -1 following the formula $\boxed{\beta_{b_j} = \begin{cases} 1 & b_j = 1 \\ -1 & b_j = 0 \end{cases}}$

$f((1, 0, 0, 1\ 0\ 0\ 1)_2) = (1, -1, -1, 1, -1, -1, 1)_2$

Observation :

$f(1, \underbrace{0, 0 \dots 0}_{\text{block of } 0}) = (1, -1, -1, -1)_2$

PROVE :

$$\boxed{(1, \underbrace{-1, -1, \dots -1}_{\text{block of } -1}) = (00\dots01)}$$

Proof of $\boxed{(1,-1,-1,\ldots -1)_2 = (0\ldots 01)_2}$

$n = \boxed{(1,-1,-1,\ldots -1)}$

$n = \underbrace{2^m - 2^{m-1}}_{2^{m-1}} - 2^{m-2} - \ldots - 2^1 - 2^0$

$= \underbrace{2^{m-1} - 2^{m-2}}_{2^{m-2}} \ldots - 2^1 - 2^0$

$\boxed{2^m - 2^{m-1} = 2^{m-1}(2-1) = 2^{m-1}}$

$2^{m-1} - 2^{m-2} = 2^{m-2}(2-1) = 2^{m-2}$

$= \underbrace{2^{m-2} - 2^{m-3}}_{2^{m-3}} \ldots 2^1 - 2^0$

$\vdots$

$= 2^1 - 2^0 = 2 - 1 = 1 = \boxed{(0\ldots 01)}$

end

$f((b_m, \ldots b_1, b_0)) = (\alpha, \beta_{b_{m-1}} \ldots \beta_{b_0})$

Each block of of binary digits $\boxed{(1,0\ldots0)}$ is transformed by f into $(1,-1\ldots-1) = \boxed{(0\ldots01)}$ and hence

$\boxed{f(b_m, \ldots b_1, b_0)_2 = (b_{m-1}, \ldots b_0, b_m)}$

$$f(b_m, b_{m-1}, \ldots b_1, b_0) = (\overset{1}{\alpha}, \beta_{m-1}, \ldots \beta_{b_0})$$

$$= (1, \beta_{b_{m-1}}, \ldots \quad \beta_{b_0})$$

via blocks

$$(1,-1,-1\ldots) = (00\ldots 1)$$

$$= (b_{m-1}, \ldots b_1, b_0, 1)$$

$$f((1, 0, 0, 1, 1, 0\,0\,1)_2) = (1, \underbrace{-1, -1}_{\text{block}}, 1, 1, -1, -1, 1)$$

$$= (0, 0, 1, 1, \underbrace{1, -1\,-1}_{\text{block}}, 1)$$

shift

$$= (0, 0, 1, 1, 0, 0, 1, 1)$$ shift.