RECCURENCE → CLOSED FORMULA → SUM

R CF S_n

GOAL

TOWER of HANOI (revisited)

R

$$\boxed{\begin{aligned} T_0 &= 0 \\ T_n &= 2T_{n-1} + 1 \end{aligned}}$$

CF QUESS

$$\boxed{T_n = 2^n - 1}$$

Divide by 2^n

$$\boxed{\begin{aligned} T_0/2^0 &= 0 \\ T_n/2^n &= \frac{T_{n-1}}{2^{n-1}} + \frac{1}{2^n} \end{aligned}}$$

$$S_n = T_n/2^n$$

USE CF

R = RS

RS

$$\boxed{\begin{aligned} S_0 &= 0 \\ S_n &= S_{n-1} + \frac{1}{2^n} \end{aligned}} \quad n>0$$

$$S_n = \frac{2^n - 1}{2^n}$$

$$S_n = \sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n}$$

SUM

$$\boxed{\sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n}}$$

$$S_n = \frac{a_1(1-q^n)}{1-q} \quad ; \quad a_1 = \frac{1}{2}, \quad q = \frac{1}{2}$$

GENERAL TECHNIQUE

for finding a CLOSED FORM

for any RECURRENCE of a TYPE: with some $n=0$ INITIAL CONDITION.

$$\boxed{R} \quad \boxed{a_n T_n = b_n T_{n-1} + c_n} \quad n \geq 1$$

BY REDUCING IT TO A CERTAIN SUM

where

$\boxed{a_n, b_n, c_n \text{ any sequences}, \quad n \geq 1}$

IDEA: Multiply R by a

smalls

$\boxed{\text{SUMMATION FACTOR } S_n, \; n \geq 1}$

(we don't know yet what this factor is, but we will find it out)

$S_n = s(n) \quad s: N \to R$

sequence

OUR R

$$\boxed{a_n T_n = b_n T_{n-1} + c_n} \qquad n \geq 1$$

s_n SUMMATION FACTOR

$$\boxed{s_n a_n T_n = s_n b_n T_{n-1} + s_n c_n} \qquad (*)$$

We WANT s_n to HAVE A PROPERTY

GOAL: GET RECURSIVE FORMULA

$$\text{(P)} \quad \boxed{s_n b_n = s_{n-1} a_{n-1}}$$

IT MEANS I TAKE a sequence s_n such that (P) holds

WE WRITE

$$\text{(S)} \quad \boxed{S_n = s_n a_n T_n}$$

$S_0 = s_0 a_0 T_0$

(as in Tower of Hanoi $S_n = \frac{1}{2^n} T_n$)

and RE-WRITE $(*)$ as (use (P))

$$\text{(RS)} \quad \boxed{S_n = S_{n-1} + s_n c_n} \qquad n \geq 1$$

$S_n = s_n b_n T_{n-1} + s_n c_n$ and $\boxed{s_n b_n = S_{n-1} a_{n-1}}$ (P)

$$S_n = s_0 a_0 T_0 + \sum_{k=1}^{n} s_k c_k$$

$$s_{n-1} a_{n-1} = s_n b_n$$

use (P)

By (S) (def)

$$S_n = a_n s_n T_n$$

we get

$$T_n = \frac{S_n}{a_n s_n}$$

$$\boxed{S_n = s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k}$$

We get $T_n = \frac{1}{a_n s_n} \cdot S_n$

S(closed)

$$\boxed{T_n = \frac{1}{s_n a_n}\left(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k\right)}$$

?

"SUM" closed formula for T_n

NEXT STEP: FIND S_n in terms of a_n, b_n, c_n

Q: HOW TO FIND

S_n FACTOR.

Evaluate P

for $n \geq 1$

$$\boxed{s_n = \frac{s_{n-1}\, a_{n-1}}{b_n}}$$

a_n, b_n are GIVEN

$$s_2 = \frac{s_1\, a_1}{b_2} = s_1 \frac{a_1}{b_2}$$

$$s_3 = \frac{s_2 \cdot a_2}{b_3} = s_1 \frac{a_1 a_2}{b_2 b_3}$$

$$s_4 = \frac{s_3\, a_3}{b_4} = s_1 \frac{a_1 a_2 a_3}{b_2 b_3 b_4}$$

BY MATH INDUCTION

$$\boxed{s_n = \underbrace{s_1}_{\text{CONSTANT}} \frac{a_1 a_2 \cdots a_{n-1}}{b_2 b_3 b_4 \cdots b_n}}$$

S

SUMMATION FACTOR

R

T_0 - given initial cond.

$$\boxed{a_n T_n = b_n T_{n-1} + c_n}$$

$n \geq 1$

a_n, b_n, c_n for $n \geq 1$

CF

$$T_n = \frac{1}{s_n a_n}\left(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k \right)$$

SUMMATION FACTOR

$$\boxed{S_n = \frac{a_1 a_2 \cdots a_{n-1}}{b_2 b_3 \cdots b_n}}$$

CONST $= S_1$

evaluate

Exaple TOWER OF HANOI

$T_0 = 0$

$T_n = 2T_{n-1} + 1$

is a case where

$$\boxed{a_n = 1,\ b_n = 2 \quad c_n = 1}$$

$S_1 = \frac{1}{2}$

$$S_n = \frac{1}{\underbrace{2 \cdots 2}_{n-1}} \cdot \underbrace{\frac{1}{2}}_{S_1} = \frac{1}{2^k}$$

$S_n = 2^{-n}$

check

$$\boxed{T_n = \frac{1}{a_n s_n}\left(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k \right)}$$

for $\boxed{s_n = 2^{-n} \;,\; a_n = 1 \;,\; b_n = 2 \;,\; c_n = 1}$ all n

$T_0 = 0$

$$T_n = \frac{1}{2^{-n}}\left(0 + \sum_{k=1}^{n} \frac{1}{2^n} \right)$$

closed SUM FORMULA

geometric

$$S_n = \frac{a_0 (q^{n+1} - 1)}{q - 1}$$

$$S_n = \frac{a_0 (1 - q^{n+1})}{1 - q}$$

$$q = \frac{1}{2}$$

[illegible]

$$\boxed{\sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n}}$$

$$T_n = 2^n \left(1 - \frac{1}{2^n} \right)$$

$$T_n = 2^n - 1$$

Closed FORMULA

QUICKSORT

HOARE 1962

The average number of comparision stops made by the QS when applied to n items in RANDOM order is

R
$$\boxed{\begin{aligned} C_0 &= 0 \\ C_n &= (n+1) + \frac{2}{n}\sum_{k=0}^{n-1} C_k \end{aligned}}$$

$C_1 = 2$, $C_2 = 5$ $C_3 = 26/3$ etc..

STEP 1 GET RID OF Σ in the reccurrence

STEP 2 Find CF (summation at least) Rec

USE PREVIOUS TECHNIQUE.

$$C_n = n+1 + \frac{2}{n}\sum_{k=0}^{n-1} C_k$$

(0)
$$\boxed{n C_n = n^2 + n + 2\sum_{k=0}^{n-1} C_k}$$ $n > 1$

$$n C_n = n^2 + n + 2\left(\sum_{k=0}^{n-2} C_k + C_{n-1}\right)$$

(1)
$$\boxed{n C_n = n^2 + n + 2\cdot\sum_{k=0}^{n-2} C_k + 2\cdot C_{n-1}}$$

Re-write (0) for $n = n-1$ we need

$$(n-1)C_{n-1} = (n-1)^2 + n - 1 + \boxed{2\sum_{k=0}^{n-2} C_k}$$

$$n^2 - 2n + 1 + n - 1$$

(2)
$$\boxed{(n-1)C_{n-1} = n^2 - n + 2\sum_{k=0}^{n-2} C_k}$$

SUBSTRUCT (2) from (1)

(3)
$$\boxed{n C_n - (n-1)C_{n-1} = 2n + 2C_{n-1}}$$

no Σ ! re-write it

7/a

We have: $\to$ we got a formula

(1) $\boxed{S_n = \frac{T_n}{2^n}}$ $\boxed{S_n = S_{n-1} + \frac{1}{2^n}}$

summation

and

(2) $\boxed{S_n = \frac{2^n - 1}{2^n}}$ by evaluating RS

(1) + (2)

$$T_n = 2^n \cdot S_n = \frac{2^n(2^n-1)}{2^n} = 2^n - 1$$

$$\boxed{T_n = 2^n - 1}$$

end.

$$\boxed{n C_n = (n-1)C_{n-1} + 2n + 2C_{n-1}}$$

NEW FORMULA $C_0 = 0$

$$= nC_{n-1} - C_{n-1} + 2n + 2C_{n-1}$$

$$= 2n + nC_{n-1} + C_{n-1}$$

FORMULA

$$\boxed{n C_n = (n+1)C_{n-1} + 2n \;,\; C_0 = 0}$$

THIS is $a_n T_n = b_n T_{n-1} + c_n$

GENERAL RECURRENCE SOLVED BEFORE!

FOR

$$\boxed{a_n = n \;,\; b_n = n+1 \;,\; c_n = 2n}$$

SUMMATION FACTOR

$$\boxed{S_n = \cancel{S_1}\frac{a_1 a_2 \dots a_{n-1}}{\cancel{b_1} b_2 \dots b_n}}$$

?

multiplied by a constant S_1?

$$S_n = \frac{1 \cdot 2 \cdots (n-1)}{\cancel{3} \cdots (n-1)\, n\, (n+1)} = \frac{2}{n(n+1)}$$

$\cdot S_1$??

$b_2 = 3$. $S_1 = \frac{2}{1 \cdot 2} = 1$ $S_1 = 1$

USE

$$\boxed{T_n = \frac{1}{s_n a_n}\left(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k\right)}$$

use this for C_n

$T_0 = C_0 = 0$

$a_n = n$

$b_n = n+1$

$c_n = 2n$

$$C_n = \frac{1}{s_n \cdot n}\left(0 + \sum_{k=1}^{n} 2k \cdot s_k\right)$$

$$s_k = \frac{2}{k(k+1)}$$

$$C_n = \frac{n(n+1)}{2n} \sum_{k=1}^{n} \frac{4k}{k(k+1)}$$

$C_0 = 0$

$$\boxed{C_n = 2(n+1) \sum_{k=1}^{n} \frac{1}{k+1}}$$

HARMONIC NUMBER H_n

$$H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$$

NAME: k-th harmonic produced by a violin string is the fundamental tone produced by a string that is $1/k$ times as long

$$\sum_{k=1}^{n} \frac{1}{k+1} = \sum_{1 \le k \le n} \frac{1}{k+1} = \sum_{1 \le k-1 \le n} \frac{1}{k}$$

put $k := k-1$

$$= \sum_{2 \le k \le n+1} \frac{1}{k} = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{1} + \frac{1}{n+1}$$

$$\boxed{\sum_{k=1}^{n} \frac{1}{k+1} = H_n - \frac{n}{n+1}}$$

$$\frac{-n-1+1}{n+1} = \; = -\frac{n}{n+1}$$

$$C_n = 2(n+1)\left(H_n - \frac{n}{n+1}\right)$$

$$= 2(n+1)H_n - \frac{2n(n+1)}{n+1}$$

CF

$$\boxed{C_n = 2(n+1)H_n - 2n, \quad C_0 = 0}$$

$C_0 = 0$, $C_1 = 1$ $C_2 = 2 \cdot 3 \cdot \frac{3}{2} - 4 = 5$ etc.

GEOMETRIC SUM REVISITED

① $$\boxed{S_n = \sum_{k=0}^{n} a x^k}$$

$$x^{k+1} = x \cdot x^k$$

x is a constant with respect to k

② Observe:

$$\boxed{\sum_{k=0}^{n} a x^{k+1} = x \sum_{k=0}^{n} a x^k}$$

PERTURBATION TECHNIQUE

Evaluate

$$\underbrace{S_n + a x^{n+1}}_{S_{n+1}} = a x^0 + \sum_{k=0}^{n} a x^{k+1}$$

$$\overset{2}{=} a + x \sum_{k=0}^{n} a x^k$$

$$\overset{1}{=} a + x S_n$$

Solve on S_n:

$$S_n - x S_n = a - a x^{n+1}$$

$$\boxed{S_n = \frac{a(1-x^{n+1})}{1-x}}$$

ANOTHER EXAMPLE OF PERTURBATION TECHNIQUE

EVALUATE

$$\boxed{S_n = \sum_{k=0}^{n} k 2^k}$$

$2^{k+1} = 2\cdot 2$

$$\boxed{S_n + (n+1)2^{n+1}} = \sum_{k=0}^{n} (k+1)2^{k+1}$$

S_{n+1}

$$= \sum_{k=0}^{n} k\cdot 2^{k+1} + \sum_{k=0}^{n} 2^{k+1}$$

geom

$$\overset{2}{=} 2\sum_{k=0}^{n} k\cdot 2^k + 2^{n+2} - 2$$

$$= \boxed{2S_n + 2^{n+2} - 2}$$

EVALUATE S_n

$$\boxed{S_n + (n+1)2^{n+1} = 2S_n + 2^{n+2} - 2}$$

$$S_n(1-2) = -(n+1)2^{n+1} + 2^{n+2} - 2 \quad /\cdot(-1)$$

$$+ \qquad - \qquad +$$

$$S_n = 2^{n+1}(n+1-2) + 2$$

$$S_n = (n-1)2^{n+1} + 2$$

CLOSED FORMULA

$$\boxed{\sum_{k=0}^{n} k\,2^k = (n-1)2^{n+1} + 2}$$

PUT $2 = x$ and REPEAT

$$\boxed{\sum_{k=0}^{n} k\,x^k = \frac{x-(n+1)x^{n+1}+n\,x^{n+2}}{(1-x)^2}}$$

Use $\displaystyle\sum_{k=0}^{n} x^{k+1} = \frac{1-x^{n+2}}{(1-x)}$

HOME.