Let's look at an example of a sum

$$\boxed{\sum_{Q(k,j)} a_{k,j}}$$

Indexes

ONE DOUBLE SEQUENCE

$N^+ = N - \{0\}$

$f : N^+ \times N^+ \to R$

$f(k,j) = a_{kj}$

for $a_{k,j} = \dfrac{1}{k-j}$

and

$$Q(k,j) = (j \le k \le n) \wedge (1 \le j \le n) \wedge (j < k)$$

$$= P(k,j) \wedge (j < k)$$

WE WANT TO EVALUATE (close f)

$$\boxed{\sum_{\substack{P(k,j)\\ j<k}} \frac{1}{k-j} = \sum_{\substack{1 \le k \le n\\ 1 \le j \le n\\ j<k}} \frac{1}{k-j} = S_n}$$

$S_1 = 0$, $S_2 = 1$ $\quad S_3 = \dfrac{1}{2-1} + \dfrac{1}{3-1} + \dfrac{1}{3-2} = 5/2$

$$\boxed{S_n = \sum_{\substack{1\le k\le n\\ 1\le j\le n\\ j<k}} \frac{1}{k-j}}$$

$$P(k,j) = (1\le k\le n) \wedge (1\le j\le n)$$

① Simplify the sum boundaries

PROVE $\Rightarrow$ $\Leftarrow$

$$\boxed{P(k,j) \wedge j<k \equiv (1\le k\le n) \wedge (1\le j<k)}$$

Evaluate

$$(1\le k\le n) \wedge (1\le j\le n) \wedge (j<k)$$

$$= \underbrace{(1\le k\le n) \wedge \overbrace{(1\le j) \wedge (j<k)}^{1\le j<k}} \wedge \underbrace{(j\le n)}$$

$$(B \wedge A) \Rightarrow B$$

$$\Downarrow$$

$$(1\le k\le n) \wedge (1\le j< k)$$

We proved

(a) $$\boxed{P(k,j) \wedge (j<k) \Rightarrow (1\le k\le n) \wedge (1\le j<k)}$$

We have to show now

(b) $$\boxed{(1\le k\le n)\wedge(1\le j<k) \Rightarrow P(k,j)\wedge(j<k)}$$

$$(1\le k\le n)\wedge(1\le j)\wedge(j<k)$$

$$=(1\le k\le n)\wedge \underbrace{(k\le n)\wedge(j<k)}_{\text{TRANSITIVITY}}\wedge(1\le j)$$

$$\Rightarrow (1\le k\le n)\wedge(j\le n)\wedge(j<k)\wedge(1\le j)$$

$$=(1\le k\le n)\wedge(1\le j\le n)\wedge(j<k)$$

(a) + (b) gives

(1) $$\boxed{P(k,j)\wedge(j<k)\equiv(1\le k\le n)\wedge(1\le j<k)}$$

BY SIMILAR METHOD we get

(2) $$\boxed{P(k,j)\wedge(j<k)\equiv(1\le j\le n)\wedge(j\le k\le n)}$$

^{123}

STEP 1 : USE (1) to EVALUATE S_n

$$S_n = \sum_{\substack{p(k,j)\\ j<k}} \frac{1}{k-j} \overset{+①}{\underset{=}{\mathrm{def}}} \sum_{k=1}^{n} \Big(\sum_{1 \le j < k} \frac{1}{k-j} \Big)$$

$\boxed{k-j := j}$

SUBSTITUTE

$$= \sum_{k=1}^{n} \Big(\sum_{1 \le k-j < k} \frac{1}{j} \Big) \rightarrow \text{condition } j \neq 0$$

Simplify the boundary

$$\boxed{1 \le k-j < k} \wedge j \neq 0 = (1 \le k-j) \wedge (k-j < k) \wedge j \neq 0$$

$$= (1+j \le k) \wedge (-j < 0) \wedge j \neq 0$$

$$= (j \le k-1) \wedge (j \geq 0) \wedge j \neq 0$$

$$= \boxed{(1 \le j \le k-1)}$$

$$\boxed{S_n} = \sum_{k=1}^{n} \Big(\sum_{j=1}^{k-1} \frac{1}{j} \Big) = \boxed{\sum_{k=1}^{n} H_{k-1}}$$

NOT CLOSED!

H_{k-1} HARMONIC

STEP 2 : USE 2 TO EVALUATE S_n

Substitute $k-j := k$

$$\boxed{S_n} = \sum_{1 \le j \le n} \left(\sum_{j < k \le n} \frac{1}{k-j} \right)$$

$$= \sum_{j=1}^{n} \left(\sum_{j < k-j \le n} \frac{1}{k} \right)$$

$$= (j < k-j) \wedge (k-j \le n)$$

$$= (0 < k) \wedge (k \le n-j)$$

$$= \sum_{j=1}^{n} \left(\sum_{k=1}^{n-j} \frac{1}{k} \right) \rightarrow \text{Harmonic } H_{n-j}$$

$$= \sum_{1 \le j \le n}^{n} H_{n-j}$$

SUBSTITUTE $n-j := j$

$$1 \le n-j \le n =$$

$$= (1 \le n-j) \wedge (n-j \le n)$$

$$= (j \le n-1) \wedge (j \ge 0)$$

$$= \sum_{1 \le n-j \le n} H_j$$

$$= \boxed{\sum_{j=0}^{n-1} H_j}$$

Still HARD TO EVALUATE !

NOT CLOSED FORM

STEP 3

$$S_n = \sum_{1 \le j < k \le n} \frac{1}{k-j}$$

SUBSTITUTE

$$\boxed{k := k + j}$$

$$= \sum_{1 \le j < k+j \le n} \frac{1}{k}$$

PROVE

① $$\boxed{1 \le j < k+j \le n = (1 \le k \le n-1) \wedge (1 \le j \le n-k)}$$

EVALUATE

② $$\boxed{S_n = \sum_{1 \le j < k+j \le n} \frac{1}{k} = \sum_{\substack{1 \le k \le n-1 \\ 1 \le j \le n-k}} \frac{1}{k}}$$

EVALUATE:

$$(1 \le j < k+j \le n) = (1 \le j) \wedge (1 \le n)$$

ⓐ

$$\wedge\ (j \le n-k) \wedge (j < k+j \le n)$$

$(k+j \le n)$

$$\overset{ⓐ}{=} (1 \le j \le n-k) \wedge (0 < k \le n-j)$$

Look at $0 < k \le n-j = \boxed{1 \le k \le n-j}$

for $j = 1, 2 \ldots n-k$ ⓑ

$1 \le k \le n-1$

$1 \le k \le n-2$

$\vdots$

$1 \le k \le k$

② we get *

$$\boxed{1 \le k \le n-1}$$

Hence

$$\boxed{(1 \le j < k+j \le n) = (1 \le j \le n-k) \wedge (1 \le k \le n-1)}$$

① Proved.

$$S_n = \sum_{\substack{1 \le k \le n-1 \\ 1 \le j \le n-k}} \frac{1}{k} = \sum_{k=1}^{n-1} \left(\sum_{j=1}^{n-k} \frac{1}{k} \right)$$

constant

$$= \sum_{k=1}^{n-1} \frac{1}{k} \sum_{j=1}^{n-k} 1$$

$$\frac{1}{k}(n-k) = \frac{n}{k} - 1$$

$$= \sum_{k=1}^{n-1} \frac{1}{k}(n-k)$$

$$= \sum_{k=1}^{n-1} \frac{n}{k} - \sum_{k=1}^{n-1} 1$$

$$= n \overbrace{\sum_{k=1}^{n-1} \frac{1}{k}}^{H_{n-1}} - (n-1)$$

$$H_{n-1} = H_n - \frac{1}{n}$$

$$= n H_{n-1} - n + 1$$

$$= n\left(H_n - \frac{1}{n}\right) - n + 1$$

$$= n H_n - 1 - n + 1$$

$$\boxed{S_n = n H_n - n}$$

(OUR) closed formula

EVALUATE (Book)

$$\boxed{S_n} \stackrel{\text{def}}{=} \sum_{k=1}^{n} \left(\sum_{1 \le j \le n-k} \frac{1}{k} \right)$$

$$= \sum_{k=1}^{n} \left(\sum_{j=1}^{n-k} \overset{\text{const}}{\frac{1}{k}} \right) = \sum_{k=1}^{n} \left(\frac{1}{k} \sum_{j=1}^{n-k} 1 \right)$$

$$= \sum_{k=1}^{n} \frac{1}{k} \cdot (n-k)$$

$$= \sum_{k=1}^{n} \left(\frac{\overset{\text{const}}{n}}{k} - 1 \right) = n \underbrace{\sum_{k=1}^{n} \frac{1}{k}}_{H_n} - \sum_{k=1}^{n} 1$$

$$= \boxed{n H_n - n}$$

$$\boxed{S_n = n H_n - n}$$

CLOSED FORMULA

BOOK FORMULA

We proved ALSO (before)

$$\boxed{S_n = \sum_{k=1}^{n} H_k}$$

We GET

REAL.

$$\boxed{\sum_{k=1}^{n} H_k = n H_n - n}$$

difficult SUM

GOAL

AND

BOOK SUM = OUR SUM

$$\boxed{\sum_{\substack{1 \le k \le n \\ 1 \le j \le n-k}} \frac{1}{k} = \sum_{\substack{1 \le k \le n-1 \\ 1 \le j \le n-k}} \frac{1}{k}}$$

$n := n-1$ in book remarks

GENERAL METHODS

A Review

PROBLEM: FIND CLOSED FORMULA FOR

$$\boxed{\square_n = \sum_{K=0}^{n} K^2}$$

Method 0: LOOK IT UP

p. 72 of CRS STANDARD ~~METHODS~~ MATHEMATICAL TABLES

$$\boxed{\square_n = \frac{n(n+1)(2n+1)}{6} \quad n \geq 0}$$

Other reference:
- HANDBOOK OF MATH. FUNCTIONS: Abramowitz, Stegun
- HANDBOOK OF INTEGER SEQUENCES, SLOANE

METHOD 1

GUESS THE ANSWER

AND PROVE BY MATH INDUCTION

GUESS:

$$\boxed{\square_n = \frac{n(n+\frac{1}{2})(n+1)}{3} = \sum_{k=0}^{n} k^2}$$

Re-write as

$$\boxed{\square_0 = 0 \quad , \quad \square_n = \square_{n-1} + n^2}$$

USE INDUCTION ASSUMPTION $\quad n := n-1$

$$3\,\square_n = (n-1)(n-1+1/2)\,n + 3n^2$$

$$= (n-1)(n-1/2)\,n + 3n^2$$

$$= n^3 - \frac{3}{2}n^2 + \frac{1}{2}n + 3n^2$$

$$= n^3 + 3/2\, n^2 + \frac{1}{2}n$$

$$= n(n^2 + 3/2\, n + 1/2)$$

$$= n(n+1/2)(n+1)$$

METHOD 2 :

PERTURB THE SUM

$$\square_n = \sum_{k=0}^{n} k^2$$

$$\square_n + (n+1)^2 = \sum_{k=0}^{n} (k+1)^2$$

$$= \sum_{k=0}^{n} (k^2 + 2k + 1)$$

$$= \sum_{k=0}^{n} k^2 + 2\sum_{k=0}^{n} k + \sum_{k=0}^{n} 1$$

$$= \square_n + 2\sum_{k=0}^{n} k + (n+1)$$

NICE CALCULATION BUT NO RESULT FOR $\square_n$! NEVERTHELESS WE GET SOMETHING :

$$(n+1)^2 = 2\sum_{k=0}^{n} k + (n+1)$$

$$\boxed{2\sum_{k=0}^{n} k = (n+1)^2 - (n+1)}$$

BONUS !

$$2\sum_{k=0}^{n} k = (n+1)(n+1-1)$$

$$\boxed{\sum_{k=0}^{n} k = \frac{n(n+1)}{2}}$$

OBSERVATION: USE K^3

BACK TO OUR PROBLEM: $\boxed{\square_n = \sum_{k=0}^{n} k^2}$

USE PERTURBATION FOR $\boxed{\boxed{\square}_n = \sum_{k=0}^{n} k^3}$ $\rightarrow$ to get $\square_n$!

WE EVALUATE (as before) as we did for $\sum_{k=0}^{n} k$

$$\boxed{\square}_n + (n+1)^3 = \sum_{k=0}^{n} (k+1)^3$$

$$= \sum_{k=0}^{n} \left(k^3 + 3k^2 + 3k + 1\right)$$

$$\boxed{(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3}$$

$$\boxplus_n + (n+1)^3 = \sum_{k=0}^{n} k^3 + 3\sum_{k=0}^{n} k^2 + 3\sum_{k=0}^{n} k + \sum_{k=0}^{n} 1$$

$\boxplus_n$ $\square_n$

$$= \boxplus_n + \boxed{3\square_n} + 3\cdot\frac{n(n+1)}{2} + (n+1)$$

GOT IT!

$$(n+1)^3 = 3\square_n + 3\cdot\frac{n(n+1)}{2} + (n+1)$$

$$3\square_n = (n+1)^3 - \frac{3n(n+1)}{2} - (n+1)$$

$$\boxed{3\square_n} = (n+1)\left((n+1)^2 - \frac{3}{2}(n\cancel{^2 n} - 1\right)$$

$$= (n+1)\left(n^2 + 2n + 1 - \frac{3}{2}n - 1\right)$$

$$= (n+1)\left(n^2 + \frac{1}{2}n\right)$$

$$= \boxed{(n+1)(n+\tfrac{1}{2})n}$$

end.

B2

$\boxed{\text{METHOD 3}}$:

$\boxed{\text{BUILD A REPERTOIRE}}$

$D_n = \sum K^2$

GENERALIZE : (1) $\boxed{\begin{aligned} R_0 &= \alpha \\ R_n &= R_{n-1} + \beta + \gamma n \end{aligned}}$

USED to $\sum_{k=0}^{n} (a+bk)$

$\alpha = \beta = a$
$\gamma = b$

TO:

(2) $\boxed{\begin{aligned} R_0 &= \alpha \\ R_n &= R_{n-1} + \beta + \gamma n + \delta n^2 \end{aligned}}$

TO evaluate $\sum (a+bk^2)$

GENERAL FORM OF FORMULA IS

$$\boxed{R_n = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\delta}$$

OBSERVE : WHEN $\boxed{\delta = 0}$ we get (1)

and $\boxed{A(n) = 1 \ , \quad B(n) = n \qquad C(n) = \frac{(n^2+n)}{2}}$

ALREADY DONE BEFORE for $\sum (a+bk)$

$\sum k^2$ generalize

$\sum(a+bk^2)$ x

$$\boxed{\begin{aligned} R_0 &= \alpha \\ R_n &= R_{n-1} + \beta + \gamma n + \delta n^2 \end{aligned}}$$

get for $\beta = a$, $\gamma = 0$, $\delta = b$

$$S_n = \sum_{k=0}^{n} (a + bk^2)$$

$$\boxed{\begin{aligned} S_0 &= a \\ S_n &= S_{n-1} + a + bk^2 \end{aligned}}$$

GENERAL CF FORMULA BECOMES

③ $$\boxed{R_n = \alpha + n\beta + \frac{(n^2+n)}{2}\gamma + D(n)\delta}$$

WE NEED to evaluate D(n)

PUT $\boxed{R_n = n^3, \text{ for all } n}$ n^3

to evaluate $\alpha, \beta, \gamma, \delta$ (if exists)

OUR RECCURENCE ② ~~becomes~~

② $$\boxed{\begin{aligned} R_0 &= \alpha \\ R_n &= R_{n-1} + \beta + \gamma n + \delta n^2 \end{aligned}}$$

becomes $R_0 = 0$ ie $\boxed{\alpha = 0}$

$$n^3 = (n-1)^3 + \beta + \gamma n + \delta n^2$$

$$\cancel{n^3} = \cancel{n^3} - 3n^2 + 3n - 1 + \beta + \gamma n + \delta n^2$$

$$0 = n^2(\delta - 3) + n(\gamma + 3) + (\beta - 1) \quad \text{ALL } n!$$

ONLY WHEN

$\boxed{\delta = 3}$, $\boxed{\gamma = -3}$, $\boxed{\beta = 1}$

OUR CF is

$$\boxed{R_n = \alpha + n\beta + \frac{n^2+n}{2}\gamma + D(n)\delta}$$

For $R_n = n^3$ and $\alpha = 0, \beta = 1, \gamma = -3, \delta = 3$

IT BECOMES

$$n^3 = 0 + n - \frac{3}{2}(n^2+n) + 3D(n)$$

$$\boxed{3D(n)} = n^3 - n + \frac{3}{2}(n^2+n)$$

$$= n^3 + \frac{3}{2}n^2 + \frac{3}{2}n - n$$

$$= n^3 + \frac{3}{2}n^2 + \frac{1}{2}n$$

$$= n\left(n^2 + \frac{3}{2}n + \frac{1}{2}\right) = \boxed{n\left(n+\frac{1}{2}\right)(n+1)}$$

$$\boxed{D(n) = \frac{n(n+\frac{1}{2})(n+1)}{3}}$$

CLOSED FORMULA

$$\boxed{R_n = \alpha + n\beta + \frac{n^2+n}{2}\gamma + \frac{n(n+\frac{1}{2})(n+1)}{3}\delta}$$

OUR SUM

$$\boxed{\Box_n = \sum_{K=0}^{n} K^2} \quad = \quad \boxed{\begin{array}{l} \Box_0 = 0 \\ \Box_n = \Box_{n-1} + n^2 \end{array}}$$

IS A SPECIAL CASE OF

$$\boxed{\begin{array}{l} R_0 = \alpha \\ R_n = R_{n-1} + \beta + \gamma n + \delta n^2 \end{array}}$$

FOR $\boxed{\alpha = 0, \ \beta = 0, \ \gamma = 0, \ \delta = 1}$

AND CLOSED FORULA

$$\boxed{R_n = \alpha + n\beta + \frac{n^2+n}{2}\gamma + \frac{n(n+\frac{1}{2})(n+1)}{3}\delta}$$

BECOMES

$$\boxed{R_n = \Box_n = \frac{n(n+1/2)(n+1)}{3}}$$

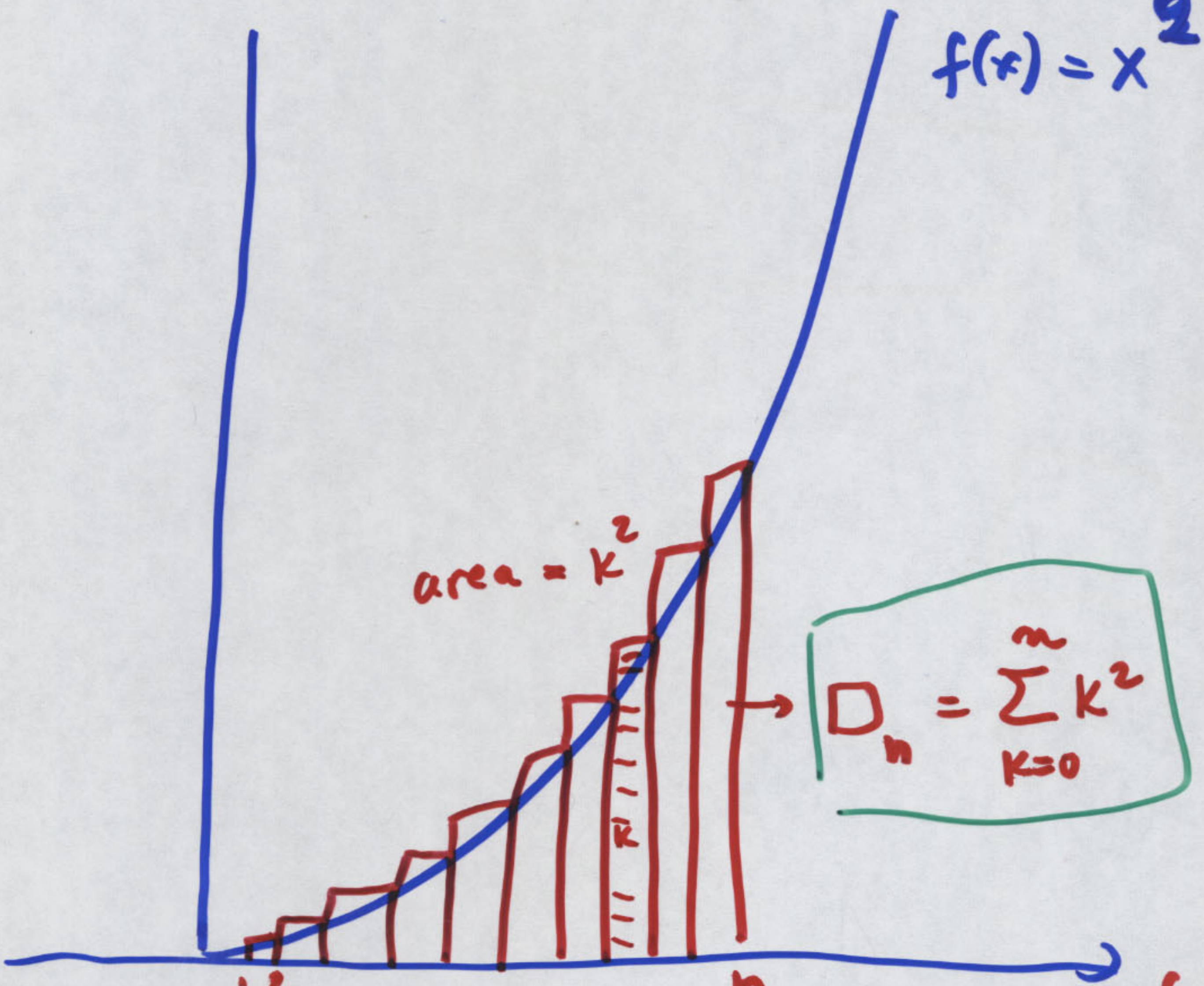
i.e

$$\boxed{\sum_{K=0}^{n} K^2 = \frac{n(n+1/2)(n+1)}{3}}$$

end

METHOD 4:

REPLACE SUMS BY INTEGRALS



$$\square_n = \sum_{k=0}^{n} k^2$$

$$\int_0^n x^2\,dx = \frac{x^3}{3}\Big|_0^n = \frac{n^3}{3}$$ ← area under the curve

$$\boxed{\square_n \sim \frac{n^3}{3}}$$

$$\boxed{\square_n = \frac{n^3}{3} + E_n}$$

error

ERROR

$$\boxed{E_n = \square_n - \frac{1}{3}n^3}$$

Want recursive formula for E_n

$$E_{n-1} = \square_{n-1} - \frac{1}{3}(n-1)^3$$

$$\boxed{E_n = \square_{n-1} + n^2 - \frac{1}{3}n^3}$$

$$= \square_{n-1} - \frac{1}{3}(n^3 - 3n^2 + 3n - 1)$$

$$\square_n = \square_{n-1} + n^2$$

$$= \square_{n-1} - \frac{1}{3}n^3 + n^2 - \frac{3}{3}n + \frac{1}{3}$$

$$= \square_{n-1} + n^2 - \frac{1}{3}n^3 + \frac{1}{3} - n$$

KNOW:

$$E_n = \square_{n-1} + n^2 - \frac{1}{3}n^3 \quad = \square_{n-1} + n^2 - \frac{1}{3}n^3 + \frac{1}{3} - n$$

ADD and subst.

$$-\frac{1}{3} + n =$$

REC

$$\boxed{E_n = E_{n-1} + n - \frac{1}{3}} \qquad E_0 = 0$$

$$\boxed{E_n = \sum_{k=1}^{n}\left(k - \frac{1}{3}\right)}$$

CLOSED FORMULA

EVALUATE

$$\boxed{E_n} = \sum_{k=1}^{n}\left(k - \frac{1}{3}\right) = \sum_{k=1}^{n} k - \frac{1}{3}\sum_{k=1}^{n} 1$$

$$= \boxed{\frac{n(n+1)}{2} - \frac{1}{3}n}$$

AND

$$\boxed{\square_n} = E_n + \frac{n^3}{3}$$

$$= \frac{n(n+1)}{2} - \frac{1}{3}n + \frac{n^3}{3}$$

$$= \frac{3n^2 + 3n - 2n + 2n^3}{6}$$

$$= \boxed{\frac{2n^3 + 3n^3 + n}{6}}$$

CHECK

$$\boxed{D_n = \frac{n(n+\frac{1}{2})(n+1)}{3}} = \frac{n^3 + \frac{1}{2}n^2 + n^2 + n}{3} = \frac{n^3 + \frac{3}{2}n^2 + \frac{1}{2}n}{3}$$

2.6 FINITE AND INFINITE CALCULUS

INFINITE CALCULUS:

Derivative OPERATOR D

$$\boxed{D f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}} = f'(x)$$

Defined for SOME functions

$$\boxed{f: R \to R}$$

called differentiable

D is called operator because it is a FUNCTION that transforms some functions into different functions

$$\boxed{D : R^R \to R^R}$$

PARTIAL

not ONTO

$f: N \to R$

FINITE CALCULUS

Difference operator Δ

$$\boxed{\Delta f(x) = f(x+1) - f(x)}$$

Δ is defined for all functions f

$$\boxed{f: R \to R}$$

Δ transforms ANY function f into another function $g(x) = f(x+1) - f(x)$

so

$$\boxed{\Delta : R^R \to R^R}$$

INTO not ONTO

Example

$\boxed{f(x) = x^m}$ $\quad f : R \to R$

$D f(x) = m x^{m-1}$

$\boxed{D(x^m) = m x^{m-1}}$ $\quad \boxed{f(x) = x^3}$

$D(x^3) = 3x^2$

WHAT ABOUT Δ?

$\Delta(x^3) = (x+1)^3 - x^3 = 3x^2 + 3x + 1$, so

$\boxed{\Delta \neq D}$

Q: Is there a function f for which $\Delta f = Df$

YES

~~But~~ there is a "new power" of x, which transforms as nicely under Δ, as x^m does under D.

FALLING FACTORIAL POWER:

$f = x^{\underline{m}}$ $f: R \to R$

DEFINITION

$$f(x) = \boxed{x^{\underline{m}} = x(x-1)(x-2)\dots(x-m+1)}$$

$m \geq 0$

We also define a $m > 0$

RISING FACTORIAL POWER:

$$f(x) = \boxed{x^{\overline{m}} = x(x+1)\dots(x+m-1)}$$

$m > 0$

$m \geq 0$

$f(n) = n^{\underline{n}}$. $f: N \to N$

Evaluate

$$\boxed{n^{\underline{n}} = n(n-1)(n-2)\dots(n-n+1) = n!}$$

Evaluate

$$\boxed{1^{\overline{n}} = 1 \cdot 2 \cdot \dots (1+n-1) = n!}$$

HOW TO DEFINE case $m=0$

$$x^{\underline{0}} = x(x-1)\dots(x+1)$$

no.

$$x^{\overline{0}} = x(x+1)\dots(x-1)$$ no.

product of no factors $= 1$

Sum of no factors $=0$

~~Evaluate~~

We define

$\boxed{x^{\underline{0}} = x^{\overline{0}} = 1}$ CASE $m=0$

$\boxed{0! = 1}$

$\boxed{x \in \mathbb{R}}$

PRODUCT OF NO FACTORS

$1^{\overline{0}} = 0! = 1 \qquad 0^{\underline{0}} = 0! = 1$

We proved:

$$\boxed{n! = n^{\underline{n}} = 1^{\overline{n}}}$$ for any $n \geq 0$

EVALUATE

$$\boxed{\Delta(x^{\underline{m}}) = (x+1)^{\underline{m}} - x^{\underline{m}}}$$

slowly next page

TO PROVE:

$$\boxed{\Delta(x^{\underline{m}}) = m\, x^{\underline{m-1}}}$$

$\boxed{D(x^m) = m x^{m-1}}$

Δ "behaves" like D on $x^{\underline{m}}$

on $x^{\underline{m}}$

Evaluate

$$\boxed{(x+1)^{\underline{m}}} = (x+1)\,x\,(x-1)\ldots(x+1-m+1)$$

$$= \boxed{(x+1)x(x-1)\ldots\ldots(x-m+2)}$$

Evaluate

$$\boxed{x^{\underline{m}}} = \boxed{x(x-1)\cdots(x-m+2)(x-m+1)}$$

Evaluate

$$\boxed{\Delta(x^{\underline{m}}) = (x+1)^{\underline{m}} - x^{\underline{m}}}$$

$$= (x+1)x(x-1)\ldots(x-m+2) - x(x-1)\ldots(x-m+2)(x-m+1)$$

$$= x(x-1)\ldots(x-m+2)\big(x+1-(x-m+1)\big)$$

$$= x(x-1)\cdots(x-m+2)\cdot m$$

$$= \boxed{m \cdot x^{\underline{m-1}}}$$

end.

Hwk Problem 7 about $x^{\overline{m}}$